DEPARTMENT OF COMMERCE

BULLETIN

OF THE

BUREAU OF STANDARDS

VOLUME 9

1913

ANALYSIS OF ALTERNATING-CURRENT WAVES BY THE METHOD OF FOURIER, WITH SPECIAL REFERENCE TO METHODS OF FACILITATING THE COMPUTATIONS

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I. INTRODUCTION.

The value of curves, showing the wave form of the electromotive force and the current in a circuit, has long been recognized, not only in the study of alternating-current phenomena, but also in precise electrical measurements, and in the design of alternating-current machinery, and various instruments have been designed for drawing such curves. These may naturally be divided into two classes—curve tracers and oscillographs. Of these, the first type is capable of giving the more precise results. The use of a curve tracer, however, presupposes steady conditions, so that the successive cycles of the wave are sensibly exact repetitions of those which have preceded. For transient phenomena the oscillograph is more suitable, and has in recent years become a powerful instrument of research. For a comprehensive and up-to-date treatment of the history and development of curve-drawing instruments, the reader is referred to the valuable treatise, "Aufnahme und Analyse von Wechselstromkurven," by E. Orlich, F. Vieweg und Sohn, Braunschweig, 1909.

Having obtained the desired curves, a simple inspection is in some cases perhaps sufficient to throw light on the problem under

consideration. More often, however, and especially where quantitative results are desired, it is necessary to make an analysis of the curve in order to realize its full value. To make such analyses, various ingenious and elaborate machines, the so-called "harmonic analyzers," have been devised, which automatically carry out the resolution of a given irregular curve into a number of component sine waves, the number depending on the number of elements included in the machine. Where a great many curves have to be analyzed, such instruments can hardly be dispensed with. Usually, however, the number of curves which have to be analyzed will hardly warrant the expense of such a machine, and one has recourse to calculation for obtaining the component waves.

The logical procedure in this case would seem to be to make use of the method of Fourier, who in his classic work on the theory of heat gave the complete solution of the problem of the resolution of a given function into a series of component sine terms. The direct use of the Fourier equations involves, however, the necessity of the formation of so many products, that the calculation is very laborious, and it is to this fact that the neglect of the Fourier method for numerical calculations is to be attributed. To avoid the difficulty, recourse has been had in a great many cases to graphical or approximate mathematical solutions, or the curve has been put aside without any analysis at all.

Realizing the greater accuracy of the Fourier method, attempts have been made from time to time to reduce the amount of labor involved in the use of Fourier's equations. Thus Perry [1] and Kintner [2] sought to simplify the calculation by the preparation of printed blank forms, in which were indicated the products to be taken in their proper places for making the summations.

A further step was taken by Rosa, who in 1897 worked out the details of a scheme of calculation, in which from 15 measured ordinates, equally spaced throughout a half wave, all odd harmonics up to and including the fifteenth can be obtained. From a consideration of the relations between the sines and cosines of supplementary and complementary angles it is easy to show that, for this case,

[1] Lond. Elect. Feb. 5, 1892, and June 28, 1895.
[2] Elect. World, 43, p. 1023; 1904.

the only functions involved are the sines of 6°, 12°, 18°, ... 90°. To facilitate the multiplication of the ordinates by these sines, Rosa prepared a multiplication table with 14 columns, in which were tabulated the calculated products of the sines of the 14 angles, other than 90°, by values of ordinates up to 50 by steps of 0.1. The calculation of the harmonics was arranged according to a definite form, each product being indicated by the number of that column, in the table, which must be entered with the values of ordinate in question.

The number of required products, when this scheme of analysis is used, is, omitting those involving sin 30° and sin 90°, 160 in number. The products omitted in this count do not of course add materially to the labor. With very little practice one can carry through a complete analysis in about 45 minutes, and this time could be reduced materially by the use of printed forms, to save the time required for writing down the numbers of the columns and the headings. This scheme, which has never been published, has been used with success for analyzing a large number of curves taken with the Rosa curve tracer; a special application was its employment for determining the correction for wave form in absolute measurements of inductance made at the Bureau of Standards.[3]

An extension of this principle of simplification of the calculation by grouping terms was carried out by Runge[4] in 1903. This method, which is described in detail below, consists in separating those products which involve the same trigonometrical function. By previously adding together the ordinates which enter into these products, the sum of the products in question is obtained with the necessity of making a single multiplication only. The paper of Runge treats the problem in an entirely general manner, and without reference to any specific problem, the number of harmonics being unrestricted. He works out in detail the scheme of calculation for the cases of 18 and 36 ordinates.

The paper of Runge does not seem to have received the attention it deserves. In 1905 S. P. Thompson[5] reviewed Runge's article, and worked out schedules for the analysis of curves from 6 and 12

[3] Rosa and Grover, This Bulletin, **1**, p. 138; 1904.
[4] Zs. für Math. und Phys., **48**, p. 443; 1903.
[5] Lond. Elect., **55**, p. 78; May 5, 1905.

measured ordinates, only odd harmonics being assumed present. The author of the present paper, after reading the article of Thompson in 1909, extended the use of the method to the case of 18 ordinates with only odd harmonics present. These three schedules are given below together with the corresponding schemes of calculation when even harmonics are present. Of the latter, that for 12 points was given by Runge; the other two have been derived by the author. During the writing of the present paper, has appeared an excellent though necessarily brief treatment of the subject of wave analysis by the method of Runge in an appendix to the "Direct and alternating current manual" of Bedell and Pierce, second edition, 1911.

Nevertheless, knowledge of the work of Runge seems to be far from general, and the Fourier analysis of alternating-current curves seems to be a thing avoided by the majority of electrical engineers. To such considerations this paper owes its origin. In it the author has endeavored to include all the necessary formulas, together with a detailed description of the methods of carrying out the calculations and checking their correctness. As a further aid toward clearness all the formulas and schemes of calculation have been illustrated by means of examples of the analyses of actual curves. A useful feature will, it is believed, be found in the multiplication tables, which allow all necessary products to be found without calculation.

For the sake of completeness, the first part of the paper includes a proof of the equations of Fourier, for the case of a finite number of terms, and with even harmonics absent. This proof follows closely the method of treatment given in Byerly's "Fourier's series and spherical harmonics."

The latter part of the paper is taken up with the consideration of some of the practical applications of curve analysis in the realm of alternating-current theory. These may perhaps appeal most strongly to the teacher, but it is hoped that they may be of use, in any case, as illustrations of general methods of attack for special problems.

Although alternating-current waves are the only cases treated here, it hardly needs to be pointed out that the methods and

results here given may, with little change, be applied to the treatment of periodic curves relating to other branches of science. This will be especially true of the analysis schedules for curves in which even harmonics are present.

The electromotive force and current waves reproduced here were all taken from originals drawn by means of the Rosa curve tracer.[6] For details of this instrument the reader is referred to the original article. Briefly, the instrument operates on a point-by-point method, the instantaneous fall of potential over a resistance, through which an alternating current is flowing being applied by means of a rotating contact maker to a potentiometer arrangement, the instantaneous emf being balanced by varying the position of a movable contact (on the potentiometer), which position can be automatically registered. The curves themselves bear witness to the faithfulness with which even insignificant irregularities of the wave may be recorded.

Finally, I wish to express my indebtedness to Prof. Rosa for the wealth of curves which he has placed at my disposal, but especially for my first interest in the subject and the advantage of an intimate knowledge of his work along this line.

II. THE FOURIER METHOD

1. DERIVATION OF FOURIER'S EQUATIONS FOR A FINITE NUMBER OF TERMS

By the method of Fourier any function may, between definite limits, be expressed as a series of sine and cosine terms in the form

$$\begin{aligned} y=f(x) =& A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x \\ &+ \ldots + A_n \sin nx + \ldots \\ &+ B_0 + B_1 \cos x + B_2 \cos 2x + B_3 \cos 3x \\ &+ \ldots + B_n \cos nx + \ldots \end{aligned} \qquad (1)$$

where $A_1, A_2, \ldots \ldots A_n, B_0, B_1, B_2, \ldots \ldots B_n$ are constants. To obtain any coefficient A_k, for example, Fourier multiplied both sides of the equation (1) by $\sin kx\, dx$ and integrated between the limits zero and X, where X is the interval through which it is

[6] Phys. Rev., 6, p. 17; 1898.

desired that the development shall hold. Since all the integrals of the forms $\int_0^X A_m \sin mx \sin kxdx$ and $\int_0^X B_m \cos mx \sin kxdx$ become severally equal to zero, the resulting equation is simply

$$\int_0^X y \sin kxdx = \int_0^X A_k \sin^2 kxdx$$

or

$$A_k = \frac{2}{X}\int_0^X y \sin kxdx$$

Similarly, the general coefficient B_k of the cosine terms is found by multiplying both members of (1) by $\cos kxdx$ and by integrating between zero and X, and we find finally,

$$B_k = \frac{2}{X}\int_0^X y \cos kxdx$$

These, which are known as the Fourier integrals, can not, in general, be evaluated, since the relation connecting y and x is usually unknown or too complicated.

The series (1) is an infinite series; only when the number of terms is infinitely great will the function be represented by the series for every value of x in the chosen interval. If n have a finite value, the series (1) will be a finite series, and the series will be equal to the function at only n points, the deviation between the function and the series being smaller the greater n is taken. Fortunately in the case of alternating current waves, the number of terms required in order that the deviations between the curve and the series may be negligible is seldom very large. Further, from the fact that well-designed alternators give waves of which the positive and negative loops are closely of the same form, it is only necessary to include terms involving the sines and cosines of odd multiples of x.

An alternating current wave may therefore in general be represented by the equation

$$\begin{aligned} y = & A_1 \sin pt + A_3 \sin 3pt + A_5 \sin 5pt + \dots \\ & + A_k \sin kpt + \dots \quad + A_{2n-1} \sin (2n-1)pt \\ & + B_1 \cos pt + B_3 \cos 3pt + B_5 \cos 5pt + \dots \\ & + B_k \cos kpt + \dots \quad + B_{2n-1} \cos (2n-1)pt \end{aligned} \tag{2}$$

where $p = 2\pi$ times the frequency.

If, therefore, ordinates of the wave be measured at $2n$ equally spaced points in the half wave, a set of $2n$ simultaneous equations of the form

$$\begin{aligned} y_0 = & B_1 + B_3 + B_5 + \dots \quad + B_k + \quad \dots \quad + B_{2n-1} \\ y_1 = & A_1 \sin \frac{\pi}{2n} + A_3 \sin \frac{3\pi}{2n} + \dots \quad + A_k \sin \frac{k\pi}{2n} + \dots \\ & + A_{2n-1} \sin \frac{(2n-1)\pi}{2n} + B_1 \cos \frac{\pi}{2n} + B_3 \cos \frac{3\pi}{2n} + \dots \\ & + B_k \cos \frac{k\pi}{2n} + \dots + B_{2n-1} \cos \frac{(2n-1)\pi}{2n} \quad \dots \\ y_m = & A_1 \sin \frac{m\pi}{2n} + A_3 \sin \frac{3m\pi}{2n} + \dots \quad + A_k \sin \frac{mk\pi}{2n} + \dots \\ & + A_{2n-1} \sin \frac{m(2n-1)\pi}{2n} + B_1 \cos \frac{m\pi}{2n} + B_3 \cos \frac{3m\pi}{2n} \\ & + \dots + B_k \cos \frac{mk\pi}{2n} + \dots \quad + B_{2n-1} \cos \frac{m(2n-1)\pi}{2n} \\ y_{2n-1} = & A_1 \sin \frac{(2n-1)\pi}{2n} + A_3 \sin \frac{3(2n-1)\pi}{2n} + \dots \\ & + A_k \sin \frac{k(2n-1)\pi}{2n} + \dots \quad + A_{2n-1} \sin \frac{(2n-1)^2\pi}{2n} \\ & + B_1 \cos \frac{(2n-1)\pi}{2n} + B_3 \cos \frac{3(2n-1)\pi}{2n} + \dots \\ & + B_k \cos \frac{k(2n-1)\pi}{2n} + \dots \quad + B_{2n-1} \cos \frac{(2n-1)^2\pi}{2n} \end{aligned} \tag{3}$$

will be obtained, which are sufficient to determine the $2n$ unknown coefficients in equation (2), and the analysis will take into account all the odd harmonics up to and including that of order $(2n-1)$. The curve represented by the equation (2) will intersect the actual curve at those points whose measured ordinates enter into equation (3) and the magnitude of the deviations between the curve and

series at the other points will give an idea as to the importance of those harmonics not taken into account by the analysis.

To obtain the value of the general coefficient A_k of the sine terms in (3), multiply both sides of the equation for y_1 by $2 \sin \frac{k\pi}{2n}$, the equation for y_2 by $2 \sin \frac{2k\pi}{2n}$, etc., and add the resulting equations. The first member of the sum becomes $2\Sigma_{m=1}^{m=2n-1} y_m \sin \frac{mk\pi}{2n}$.

In the second member the general coefficient A_r (where r can have any value except k) is multiplied by the factor

$$2\Sigma_{m=1}^{m=2n-1} \sin \frac{mr\pi}{2n} \sin \frac{mk\pi}{2n} = 2\Sigma_{m=1}^{m=2n-1}\left[\cos \frac{m(r-k)\pi}{2n} - \cos \frac{m(r+k)\pi}{2n}\right]$$

Making use of the lemma [7]

$$\cos x + \cos 2x + \cos 3x + \ldots + \cos (s-1)x + \cos sx = -\tfrac{1}{2} + \tfrac{1}{2}\,\frac{\sin (2s+1)\frac{x}{2}}{\sin \frac{x}{2}} \tag{4}$$

and remembering that $(r-k)$ and $(r+k)$ are even integers, it is easy to show that the factor by which A_r is multiplied is equal to zero.

The coefficient B_r (where r may have any value, not excluding k) is

$$2\Sigma_{m=1}^{m=2n-1} \sin \frac{mr\pi}{2n} \cos \frac{mk\pi}{2n} = 2\Sigma_{m=1}^{m=2n-1}\left[\sin \frac{m(r-k)\pi}{2n} + \sin \frac{m(r+k)\pi}{2n}\right]$$

and making use of the lemma [7]

$$\sin x + \sin 2x + \sin 3x + \ldots + \sin (s-1)x + \sin sx = \tfrac{1}{2}\,\frac{\sin \frac{sx}{2} \sin (s+1)\frac{x}{2}}{\sin \frac{x}{2}} \tag{5}$$

the factor by which B_r is multiplied is also seen to be equal to zero.

[7] Byerly's Fourier's Series and Spherical Harmonics, p. 32.

There consequently remains in the second member only one term

$$2A_k\Sigma_{m=1}^{m=2n-1} \sin \frac{2mk\pi}{2n} = A_k\left[(2n-1) - \Sigma_{m=1}^{m=2n-1} \cos \frac{mk\pi}{n}\right]$$

which by (4) reduces to $2nA_k$, and we have finally

$$2nA_k = 2\Sigma_{m=1}^{m=2n-1} y_m \sin \frac{mk\pi}{2n}$$

or (6)

$$A_k = \frac{1}{n}\Sigma_{m=1}^{m=2n-1} y_m \sin \frac{mk\pi}{2n}$$

To obtain B_k we multiply both sides of the equation for y_0 by 2, the equation for y_1 by $2 \cos \frac{k\pi}{2n}$, the equation for y_2 by $2 \cos \frac{2k\pi}{2n}$, etc., and add these equations. The first member of the resulting equation is $2\Sigma_{m=0}^{m=2n-1} y_m \cos \frac{mk\pi}{2n}$.

In the second member, the coefficient of A_r is

$$2\,\Sigma_{m=1}^{m=2n-1} \sin \frac{mr\pi}{2n} \cos \frac{mk\pi}{2n},$$

which we have already shown reduces to zero.

The general coefficient B_r, where r does not have the value k, is affected by the factor

$$2\Sigma_{m=0}^{m=2n-1} \cos \frac{mk\pi}{2n} \cos \frac{mr\pi}{2n}$$
$$= 2 + \Sigma_{m=1}^{m=2n-1}\left[\cos \frac{m(r+k)\pi}{2n} + \cos \frac{m(r-k)\pi}{2n}\right]$$

which by use of the lemma (4) can be shown to be equal to zero.

There remains therefore in the second member, one term only, $2B_k\Sigma_{m=0}^{m=2n-1} \cos^2 \frac{mk\pi}{2n} = 2nB_k$ and the value of B_k follows at once.

It has been shown, therefore, that the alternating wave can be represented by the finite series (2) at $2n$ points of the half wave, and that the coefficients in (2) are capable of calculation from the $2n$ ordinates of these points, by the simple relations

$$A_k = \frac{1}{n}\Sigma_{m=0}^{m=2n-1} y_m \sin \frac{mk\pi}{2n}, \quad B_k = \frac{1}{n}\Sigma_{m=0}^{m=2n-1} y_m \cos \frac{mk\pi}{2n} \qquad (7)$$

The coefficients of the sine terms in (2) are therefore found by taking the averages of the measured ordinates of the curve, each ordinate having been multiplied by the sine of an appropriate multiple of that angle which indicates the position of the ordinate in question. Similarly, the coefficients B_k of the cosine terms are found by averaging the products of the measured ordinates y_m and the cosines of the same multiples of the abscissas. For example, if 12 ordinates are measured with abscissas equally spaced 15° apart, to find the coefficient B_3 we are directed by equation (7) to multiply y_0 by cos 0°, y_1, by cos $3\times 15°$, y_2 by cos $2\times 3\times 15°$, etc., to add these 12 products, and to divide the sum by 6.

Simple as are the operations indicated in (7) for finding the coefficients in the Fourier's series development for an alternating current wave, it is, however, evident that the amount of labor involved in evaluating the coefficients must increase rapidly with the number of harmonics taken into account; that is, with the number of terms included in the series. If any considerable number of harmonics are to be found, it will become necessary to systematize the calculation in order to avoid confusion and to guard against error.

2. RUNGE'S METHOD OF GROUPING

Of the methods which have been suggested, the one which best serves the purpose is that of Runge; it is used in the methods of analysis considered below.

We note, first, that it it is an advantage to use an even number of measured ordinates per half wave, rather than an odd number. Not only is the necessity avoided for the calculation of one of the coefficients from an extra set of ordinates, as remarked above in the case of A_{15}, but it has the further advantage that the grouping of terms involving the sines of common angles can be further extended, and the calculation of certain higher harmonics can be made to depend in a simple manner on the calculation of the lower harmonics.

In general, since k is an odd number

$$\begin{aligned}
&\sin\frac{mk\pi}{2n}=\sin\,(2n-m)\frac{k\pi}{2n}, \qquad &&\cos\frac{mk\pi}{2n}=-\cos\,(2n-m)\frac{k\pi}{2n}\\
&\sin k\frac{m\pi}{2n}=\pm\sin\,(2n-k)\frac{m\pi}{2n}, \qquad &&\cos k\frac{m\pi}{2n}=\mp\cos\,(2n-k)\frac{m\pi}{2n}
\end{aligned} \tag{8}$$

In those cases where the double sign appears, the upper sign holds for odd values of m and the lower sign for even values of m.

The first two of the equations (8) show that in the fundamental equations (7) the measured ordinates y_m and y_{2n-m} are to be multiplied by the sine of the same angle, and, excepting for the algebraic sign, by the same cosine. Therefore, by adding y_m and y_{2n-m} before multiplying by the sine, and by taking the difference of y_m and y_{2n-m} before multiplying by the cosine, the number of products to be obtained is halved. In this method, therefore, we obtain at the start the quantities $s_m = y_m + y_{2n-m}$ and $d_m = y_m - y_{2n-m}$, the sums and differences of complementary ordinates.

From the last two equations of (8) it is evident that the products entering into the calculation of the coefficients A_k and B_k, on which the amplitude of the kth harmonic depends, are also required in the calculation of the $(2n-k)$th harmonic. It is not difficult to show that if we separate those products which involve even ordinates from those involving the odd ordinates, and take the respective sums S_e and S_o of the sine products involving the s_{2m} and s_{2m+1} and the sums D_e and D_o of the products involving the d_{2m} and d_{2m+1}, respectively, then the coefficients A_k, A_{2n-k}, B_k, B_{2n-k} are given by the following relations:

$$\begin{aligned} A_k &= S_o + S_e, & A_{2n-k} &= S_o - S_e \\ B_k &= D_o + D_e, & B_{2n-k} &= D_o - D_e \end{aligned} \tag{9}$$

The coefficients may consequently be separated into two complementary groups. For those harmonics whose order k is a factor of $2n$, the calculation becomes yet simpler. Putting $\frac{k}{2n} = \frac{1}{\nu}$, where ν is an integer, we have

$$\sin \frac{mk\pi}{2n} = \sin \frac{m\pi}{\nu} = -\sin \frac{m+\nu}{\nu}\pi = \sin \frac{m+2\nu}{\nu}\pi,$$

$$\cos \frac{mk\pi}{2n} = \cos \frac{m\pi}{\nu} = -\cos \frac{m+\nu}{\nu}\pi = \cos \frac{m+2\nu}{\nu}\pi, \text{ etc.,}$$

and if, further, m is exactly divisible by ν, $\sin \frac{mk\pi}{2n} = \pm 1$, $\cos \frac{mk\pi}{2n} = \text{zero}$, and for values of m half as great, $\sin \frac{mk\pi}{2n} = \pm 1$, $\cos \frac{mk\pi}{2n} = \text{zero}$. A considerable number of terms, therefore, involve the factor zero or unity, or at least the sine or cosine of the same angle.

III. SCHEDULES FOR CARRYING OUT ANALYSES

1. ARRANGEMENT OF CALCULATIONS

From these considerations are derived the following schedules based, respectively, upon systems of 6, 12, or 18 equally spaced measured ordinates. Blank forms can easily be prepared, to save clerical labor, if many calculations have to be made.

TABLE 1

Six Point Schedule

Measured ordinates	Sums	Diffs.		Sine terms: 1st and 5th		Sine terms: 3d	Cosine terms: 1st and 5th		Cosine terms: 3d
y_0		d_0							
y_1 y_5	s_1	d_1	sin 30°	s_1			d_2		
y_2 y_4	s_2	d_2	" 60°		s_2			d_1	
y_3	s_3		" 90°	s_3		s_1-s_3	d_0		d_0-d_2
			sums	So	Se	S	Do	De	D
				$A_1=\frac{So+Se}{3}$		$A_3=\frac{S}{3}$	$B_1=\frac{Do+De}{3}$		$B_3=\frac{D}{3}$
				$A_5=\frac{So-Se}{3}$			$B_5=\frac{Do-De}{3}$		

In each of these schedules the measured ordinates are first written down in two columns in the order indicated. In the next two columns appear the sums s_m of the ordinates, found by adding those in the same row, and the differences d_m of the same ordinates. In the fifth column are indicated the trigonometric functions which enter into the calculation. The rest of the schedule indicates in an abbreviated form what products are to be formed, the convention being adopted that each quantity s_m or d_m is to be be multiplied by the sine of the angle which appears in the same

row at the left. For example, in the 6-point schedule, we are to take the product of s_1 and sin 30° in one case, and in another the product of d_2 and sin 30°. It is also to be noticed that each product which is involved in the calculation of any coefficient, stands in the left or right-hand column, according as it depends

TABLE 2

Twelve Point Schedule

Measured ordinates		Sums	Diffs.		Sine terms					
					1st and 11th		3d and 9th		5th and 7th	
y_0		-	d_0							
y_1	y_{11}	s_1	d_1	sin 15°	s_1				s_5	
y_2	y_{10}	s_2	d_2	" 30°		s_2				s_2
y_3	y_9	s_3	d_3	" 45°	s_3		σ_1		$-s_3$	
y_4	y_8	s_4	d_4	" 60°		s_4				$-s_4$
y_5	y_7	s_5	d_5	" 75°	s_5				s_1	
y_6		s_6		" 90°		s_6		σ_2		s_6
					So^{I}	Se^{I}	So	Se	So^{II}	Se^{II}
		$\sigma_1 = s_1 + s_3 - s_5$	$\delta_1 = d_1 - d_3 - d_5$		$A_1 = \frac{So^{I}+Se^{I}}{6}$		$A_3 = \frac{So+Se}{6}$		$A_5 = \frac{So^{II}+Se^{II}}{6}$	
		$\sigma_2 = s_2 - s_6$	$\delta = d_0 - d_4$		$A_{11} = \frac{So^{I}-Se^{I}}{6}$		$A_9 = \frac{So-Se}{6}$		$A_7 = \frac{So^{II}-Se^{II}}{6}$	

	Cosine terms					
	1st and 11th		3d and 9th		5th and 7th	
sin 15°		d_5				d_1
" 30°	d_4				d_4	
" 45°		d_3		δ_1		$-d_3$
" 60°	d_2				$-d_2$	
" 75°		d_1				d_5
" 90°	d_0		δ_2		d_0	
	Do^{I}	De^{I}			Do^{II}	De^{II}
	$B_1 = \frac{Do^{I}-De^{I}}{6}$		$B_3 = \frac{\delta_2+\delta_1}{6}$		$B_5 = \frac{Do^{II}+De^{II}}{6}$	
	$B_{11} = \frac{Do^{I}=De^{I}}{6}$		$B_9 = \frac{\delta_2-\delta_1}{6}$		$B_7 = \frac{Do^{II}-De^{II}}{6}$	

upon odd or even ordinates. For example, in the calculation of the fifth and seventh harmonics in the 12-point schedule, we have to form the products s_5 sin 15°, $-s_3$ sin 45°, and s_1 sin 75° and take

their sum obtaining the quantity S_0''. Similarly, the quantity S_e'' is found as the sum of the products s_2 sin 30°, $-s_4$ sin 60°, and s_6 sin 90°. In the case of the 12-point schedule the coefficients

TABLE 3

Eighteen Point Schedule

Measured ordinates	Sums	Diffs.		Sine terms: 1st and 17th		3d and 15th		5th and 13th		7th and 11th		9th
y_0		d_0										
y_1 y_{17}	s_1	d_1	sin 10°	s_1				$-s_7$		$-s_5$		
y_2 y_{16}	s_2	d_2	" 20°		s_2				$-s_4$		$-s_8$	
y_3 y_{15}	s_3	d_3	" 30°	s_3		σ_1		s_3		$-s_3$		
y_4 y_{14}	s_4	d_4	" 40°		s_4				s_8		s_2	
y_5 y_{13}	s_5	d_5	" 50°	s_5				s_1		s_7		
y_6 y_{12}	s_6	d_6	" 60°		s_6		σ_2		$-s_6$		s_6	
y_7 y_{11}	s_7	d_7	" 70°	s_7				$-s_5$		s_1		
y_8 y_{10}	s_8	d_8	" 80°		s_8				s_2		$-s_4$	
y_9	s_9		" 90°	s_9		σ_3		s_9		$-s_9$		Σ
$\sigma_1=s_1+s_5-s_7$				So^{I}	Se^{I}	So^{II}	Se^{II}	So^{III}	Se^{III}	So^{IV}	Se^{IV}	
$\sigma_2=s_2+s_4-s_8$ $\Sigma=\sigma_1-\sigma_3$				$A_1=\frac{So^{I}+Se^{I}}{9}$		$A_3=\frac{So^{II}+Se^{II}}{9}$		$A_5=\frac{So^{III}+Se^{III}}{9}$		$A_7=\frac{So^{IV}+Se^{IV}}{9}$		$A_9=\frac{\Sigma}{9}$
$\sigma_3=s_3-s_9$				$A_{17}=\frac{So^{I}-Se^{I}}{9}$		$A_{15}=\frac{So^{II}-Se^{II}}{9}$		$A_{13}=\frac{So^{III}-Se^{III}}{9}$		$A_{11}=\frac{So^{IV}-Se^{IV}}{9}$		

		Cosine terms: 1st and 17th		3d and 15th		5th and 13th		7th and 11th		9th
	sin 10°	d_8				$-d_2$		d_4		
$\delta_0=d_0-d_6$	" 20°		d_7				$-d_5$		d_1	
	" 30°	d_6		δ_2		d_6		d_6		
$\delta_1=d_1-d_5-d_7$	" 40°		d_5				d_1		$-d_7$	
$\delta_2=d_2-d_4-d_8$	" 50°	d_4				d_8		$-d_2$		
	" 60°		d_3		δ_1		$-d_3$		$-d_3$	
$\Delta=\delta_0-\delta_2$	" 70°	d_2				$-d_4$		$-d_8$		
	" 80°		d_1				d_7		d_5	
	" 90°	d_0		δ_0		d_0		d_0		Δ
		Do^{I}	De^{I}	Do^{II}	De^{II}	Do^{III}	De^{III}	Do^{IV}	De^{IV}	
		$B_1=\frac{Do^{I}+De^{I}}{9}$		$B_3=\frac{Do^{II}+De^{II}}{9}$		$B_5=\frac{Do^{III}+De^{III}}{9}$		$B_7=\frac{Do^{IV}+De^{IV}}{9}$		$B_9=\frac{\Delta}{9}$
		$B_{17}=\frac{Do^{I}-De^{I}}{9}$		$B_{15}=\frac{Do^{II}-De^{II}}{9}$		$B_{13}=\frac{Do^{III}-De^{III}}{9}$		$B_{11}=\frac{Do^{IV}-De^{IV}}{9}$		

A_3, A_9, B_3, and B_9 involve several of the differences d_m and the sums s_m in each of the products. Thus, in the case of the 12-point

schedule, the quantity $\sigma_1 = s_1 + s_3 - s_5$ has to be multiplied by sin 45° as also the quantity $\delta_1 = d_1 - d_3 - d_5$. These schedules will be further explained and illustrated by numerical examples below.

To aid in making the numerical calculations, there have been tabulated (Appendix A) the products of all the sines which enter into these schedules by all the whole numbers up to 100. These products are carried out to three decimal places. Interpolation is accomplished by means of the same table by simply shifting the decimal point. Thus, any product of an ordinate of four figures may be found by entering the table twice and making a simple addition. For example, the product of the ordinate 74.39 by sin 70° is found by entering column 7 of the table with argument 74 and 39. The numbers found are 69.538 and 36.648. The required product is therefore 69.538 + .366 = 69.904. Those products which involve sin 30° and sin 90° will of course be obtained without the aid of the table.

In work of this kind some check on the accuracy of the numerical work is almost indispensable. Fortunately, such a check may be made without any considerable amount of labor. The equations below give sufficient relations between the coefficients and the measured ordinates to establish the correctness of the values of the coefficients derived by calculations.

2. CHECKS ON THE ACCURACY OF THE CALCULATIONS

Check on the 6-point Analysis

$$\begin{aligned} y_0 &= (B_1 + B_5) + B_3 \\ y_3 &= (A_1 + A_5) - A_3 \\ s_2 &= 2(A_1 - A_5) \sin 60° \\ d_1 &= 2(B_1 - B_5) \sin 60° \end{aligned} \tag{10}$$

Check on the 12-point Analysis

$$\begin{aligned} y_0 &= (B_1 + B_{11}) + (B_3 + B_9) + (B_5 + B_7) \\ y_6 &= (A_1 - A_{11}) - (A_3 - A_9) + (A_5 - A_7) \\ s_3 &= 2 \sin 45°.[(B_1 - B_{11}) - (B_3 - B_9) - (B_5 - B_7)] \\ d_3 &= 2 \sin 45°.[(A_1 + A_{11}) + (A_3 + A_9) - (A_5 + A_7)] \\ d_2 &= 2 \sin 60°.[(B_1 + B_{11}) - (B_5 + B_7)] \\ s_4 &= 2 \sin 60°.[(A_1 - A_{11}) - (A_5 - A_7)] \end{aligned} \tag{11}$$

Of the equations (11) the first gives a check on the sums of the complementary B coefficients, while the second gives an indication of the accuracy of the differences of the complementary A coefficients. If the third and fourth of the equations are also satisfied the check is complete, since when the sums and differences of the complementary coefficients have been shown to be correct the individual coefficients must necessarily be correct. In case of the failure of the calculated values to satisfy any one of the equations, it is difficult to determine from the equations wherein the trouble lies. Herein the check for the 12-point analysis is not so convenient as that for the 6-point analysis. The last two equations in (11) have accordingly been appended to aid in such a case, since they do not contain the coefficients A_3, A_9, B_3, or B_9.

Check on the 18-point Analysis

$$\left.\begin{aligned}
y_0 &= (B_1+B_{17})+(B_3+B_{15})+(B_5+B_{13})+(B_7+B_{11})+B_9\\
y_9 &= (A_1+A_{17})+(A_5+A_{13})+A_9-(A_3+A_{15})-(A_7+A_{11})\\
d_3 &= 2\sin 60°.[(B_1-B_{17})-(B_5-B_{13})-(B_7-B_{11})]\\
s_6 &= 2\sin 60°.[(A_1-A_{17})-(A_5-A_{13})+(A_7-A_{11})]
\end{aligned}\right\} \quad (12)$$

The first two of the equations (12) should first be applied. If these are satisfied, the sums of all the complementary coefficients are correct, which shows that the quantities S_o and D_o have been correctly computed. Similarly, the third equation is a check on the accuracy of the calculation of the quantities D_e upon which the differences of the complementary B coefficients depend, while the fourth equation gives the corresponding check on the quantities S_e. The system of control shown in equations (10), (11), and (12) gives not only a complete check on the calculation of the coefficients, but in case of error serves to indicate within comparatively narrow limits what part of the work needs to be examined for error. The only combinations of coefficients which do not enter into the check equations (12) are the differences of the third and fifteenth harmonics, and the calculation of these is so simple that it is best to take the few moments necessary to repeat the calculation of these quantities. Since the products involved in $A_3-A_{15}=S_e''$ and $B_3-B_{15}=D_e''$ contain sin 30° and sin 90° only,

the chances of error in taking the products is no greater than that in applying any of the equations (12).

It is recommended that every analysis be checked. In spite of the fact that simple arithmetical operations only enter into the analysis, errors very easily slip in, which are readily detected on carrying through the check, and in any case the small amount of time necessary to apply the check is amply repaid by the added confidence thus lent to the results of the analysis.

3. CALCULATION OF THE AMPLITUDE AND PHASE OF THE HARMONICS

Having obtained and checked the values of the coefficients A_k and B_k in the Fourier equation (2), it still remains to calculate from these coefficients the amplitudes and phase relations of the different harmonics. For this purpose, we designate by C_k the amplitude of the kth harmonic and by θ_k the difference of phase between this harmonic and the arbitrary phase of reference, which is that of the ordinate chosen as y_0. Putting $A_k = C_k \cos \theta_k$ and $B_k = C_k \sin \theta_k$ we have in general

$$\begin{aligned} A_k \sin kpt + B_k \cos kpt &= C_k \sin (kpt - \theta_k) \\ &= C_k \sin k(pt - \psi_k) \end{aligned} \tag{13}$$

where

$$C_k = \sqrt{A_k^2 + B_k^2}$$

$$\tan \theta_k = \frac{-B_k}{A_k} \qquad \psi_k = \frac{\theta_k}{k}$$

The quadrant of θ_k is uniquely determined from the consideration that the algebraic signs of the numerator and denominator in the equation for $\tan \theta_k$ are respectively those of $\sin \theta_k$ and $\cos \theta_k$. The phase of any harmonic C_k with respect to the fundamental, expressed in terms of the period of the latter, is given by $(\theta_1 - \psi_k)$, or we may displace the origin so as to refer the whole curve to the zero point of the fundamental by making $\theta_1 = 0$, and subtracting θ_1 from the value of ψ_k for each harmonic. The angle θ_k expresses the difference in phase between the harmonic and the point of reference, but in terms of the period of the harmonic itself. It is convenient to use θ_k or ψ_k according to circumstances.

4. CHOICE OF THE SCHEDULE TO BE USED

Using the schedules detailed above, the analysis of a curve may be easily and expeditiously carried out. As an indication of the amount of labor required, the number of entries which must be made in the multiplication table is as follows: 6-point, 2; 12-point, 14; and for the 18-point schedule, 40. The additions and subtractions require, in any given case, about half as much time as the multiplications. The total length of time necessary for calculating the coefficients may be estimated as less than 10 minutes for a 6-point analysis, and about three-quarters of an hour for an 18-point analysis.

The question naturally arises as to the choice of schedule in any given case. This depends, of course, both on the nature of the curve and on the precision desired. For example, if the curve be free from ripples of short period, and closely approximates in appearance to a sine wave, it will be sufficient to take into account nothing higher than the fifth harmonic, and the 6-point schedule will be chosen. On the contrary, if the curve be more or less irregular, and especially if the sinuosities are of short period compared with the fundamental, it will be necessary to carry out the analysis to include higher harmonics. However, it is not always easy to judge what higher-order terms should be looked for. It very often occurs that several harmonics may be present to nearly an equal degree, in which case the curve will show a peak wherever the maxima of these harmonics come nearly together, and a trough in those regions where they reach their negative maxima simultaneously. In such cases only a very imperfect idea can be gained, by mere inspection, of the number and nature of the harmonics present. In the case of slotted armatures one can predict with a good deal of certainty that certain harmonics should be present in the emf. wave. For example, if an armature have 13 slots per pair of poles in the field, it is probable that the thirteenth harmonic will be present to a notable extent in the emf. wave; if there be 12 slots per pair of poles, then the eleventh and thirteenth harmonics are to be expected.

To be absolutely certain, in a given case, that no harmonic of higher order than those included in the analysis is present to an appre-

ciable extent, it is necessary to compute from the calculated coefficients the values of the ordinates of the curve for points other than those used in the analysis, and to compare the values thus obtained with the actual measured values at these points. For this test enough points should be included to avoid the possibility of considering only those places where the harmonics to be detected nearly annul each other, or are near their minimum values. The matter is somewhat complicated by the fact that with harmonics present of higher order than those included in the analysis, the value of the highest harmonic actually calculated comes out, not with its true value, but of such a magnitude as to correct for the effects of yet higher harmonics at the phases of the fundamental ordinates.

The calculation of the ordinates of points intermediate to those included in the analysis, using the values of the harmonics derived by an analysis, may be simplified by properly choosing the ordinates for which the calculation is to be made. The derivation of the schemes of calculation given in the following table is a simple matter; the nomenclature is the same as in the preceding pages.

The phases of the ordinates which may be calculated by Table 4 are indicated by subscripts. For example, the ordinate y_{50}, in the middle section of the table lies 50° away from the fundamental ordinate y_0 used in the analysis. The table indicates that the value of y_{50} which is consistent with the results of the analysis is to be found by performing certain operations on the coefficients which have been found by the analysis. Thus, A_3 is to be multiplied by sin 30°, $-A_5$ by sin 70°, etc. These products are arranged in two groups whose sums M_5 and N_5 are then to be found. The sum (M_5+N_5) gives the required value of y_{50}, and the difference (M_5-N_5), the value of its complementary ordinate y_{130}. On comparing these calculated values with the actual ordinates of the curve, an idea of the adequacy of the analysis is obtained.

It will sometimes occur that more or less complete resonance of a harmonic of high frequency will exist, and will give the wave a rippled appearance such as to allow the order of the resonant harmonic to be established with certainty. For example, if it is practically certain from the frequency of the ripples that a component emf. of a frequency 21 times that of the fundamental is

present, the magnitude and phase of this harmonic may be obtained by measuring 22 equidistant points per half wave, and

TABLE 4

For the Calculation of the Ordinates of Points Intermediate to Those Used in the Analysis

SIX-POINT SCHEDULE

sin 15°	A_1	B_5			A_5	B_1
" 45°	A_3	B_3	$(A_1+A_3-A_5)$	$(B_1-B_3-B_5)$	$-A_3$	$-B_3$
" 75°	A_5	B_1			A_1	B_5
Sums	M_1	N_1	M_2	N_2	M_3	N_3
	$y_{15}=M_1+N_1$		$y_{45}=M_2+N_2$		$y_{75}=M_3+N_3$	
	$y_{165}=M_1-N_1$		$y_{135}=M_2-N_2$		$y_{105}=M_3-N_3$	

TWELVE-POINT SCHEDULE

sin 10°	A_1			$-B_5$		(B_7+B_{11})	$-(A_7+A_{11})$		$-A_5$			B_1
" 20°		(B_7-B_{11})	A_1		$-A_5$			$-B_5$		B_1	$-(A_7+A_{11})$	
" 30°	A_3			B_3		$-B_3$	A_3		$-A_3$			$-B_3$
" 40°		B_5	(A_7-A_{11})		A_1			B_1		$-(B_7-B_{11})$	A_5	
" 50°	A_5			$-(B_7+B_{11})$		B_1	A_1		(A_7+A_{11})			B_5
" 60°		B_3	A_3		A_3			$-B_3$		$-B_3$	$-A_3$	
" 70°	(A_7+A_{11})			B_1		$-B_5$	$-A_5$		A_1			$-(B_7+B_{11})$
" 80°		B_1	A_5		$-(A_7-A_{11})$			(B_7-B_{11})		B_5	A_1	
" 90°	A_9			$-B_9$		B_9	A_9		$-A_9$			B_9
Sums	M_1	N_1	M_2	N_2	M_4	N_4	M_5	N_5	M_7	N_7	M_8	N_8
M+N	y_{10}		y_{20}		y_{40}		y_{50}		y_{70}		y_{80}	
M−N	y_{170}		y_{160}		y_{140}		y_{130}		y_{110}		y_{100}	

EIGHTEEN-POINT SCHEDULE

sin 15°	a_1	b_3			a_3	b_1
" 45°	a_2	b_2	α	β	$-a_2$	$-b_2$
" 75°	a_3	b_1			a_1	b_3
Sums	M_1	N_1	α	β	M_3	N_3
	$y_{15}=M_1+N_1$		$y_{45}=\alpha+\beta$		$y_{75}=M_3+N_3$	
	$y_{165}=M_1-N_1$		$y_{135}=\alpha-\beta$		$y_{105}=M_3-N_3$	

where

$a_1=A_1+A_{11}-A_{13}$	$b_1=B_1-B_{11}-B_{13}$	$\alpha=a_1+a_2-a_3$
$a_2=A_3+A_9-A_{15}$	$b_2=B_3-B_9-B_{15}$	$\beta=b_1-b_2-b_3$
$a_3=A_5+A_7-A_{17}$	$b_3=B_5-B_7-B_{17}$	

calculating the values of A_{21} and B_{21} directly by means of equation (7). It is well in such a case to recalculate the harmonic from

other sets of 22 ordinates, since it is hard to be sure that other harmonics of nearly the same frequency are not also present. See example 3.

Evidently, in case of doubt, such tests require about as much time as the analysis itself, so that it will generally be well to err on the safe side, and select that schedule of analysis, which will certainly include all those harmonics which are reasonably to be expected. A second analysis using the same schedule, but with a different set of ordinates will if in good agreement prove that the neglected harmonics are negligible. Fortunately, in the great majority of cases, harmonics of order higher than the fifteenth are of rare occurrence, and only in the case of resonance will the value of these upper harmonics be appreciable.

IV. EXAMPLES OF THE USE OF THE ANALYSIS SCHEDULES

EXAMPLE 1. ANALYSIS OF A CURVE BY THE SIX-POINT SCHEDULE

As an example of the application of the 6-point schedule we may take the curve A, Fig. 1, which shows the current wave in a highly inductive circuit. This curve approximates very closely a simple sine wave in its appearance, and no appreciable harmonics of high order are to be expected.

The fundamental ordinates used, which are the means of those of two consecutive half waves are as follows:

$y_0 = -1.3$			$d_0 = -1.3$
$y_1 = 15.1$	$y_5 = 17.3$	$s_1 = 32.4$	$d_1 = -2.2$
$y_2 = 28.0$	$y_4 = 28.85$	$s_2 = 56.85$	$d_2 = -0.85$
$y_3 = 32.35$		$s_3 = 32.35$	

The average deviations of the pairs of measured ordinates from their means was about 0.4 of a unit on the same scale, which difference is mainly due to the fact that the zero line of the paper was slightly displaced from the actual line of zero electromotive force.

The analysis is given in full in Table 5, together with the check given in equations (10). Italic type indicates that part of the work which could be saved by the use of a printed form. In the column at the extreme left are given the numbers of the rows.

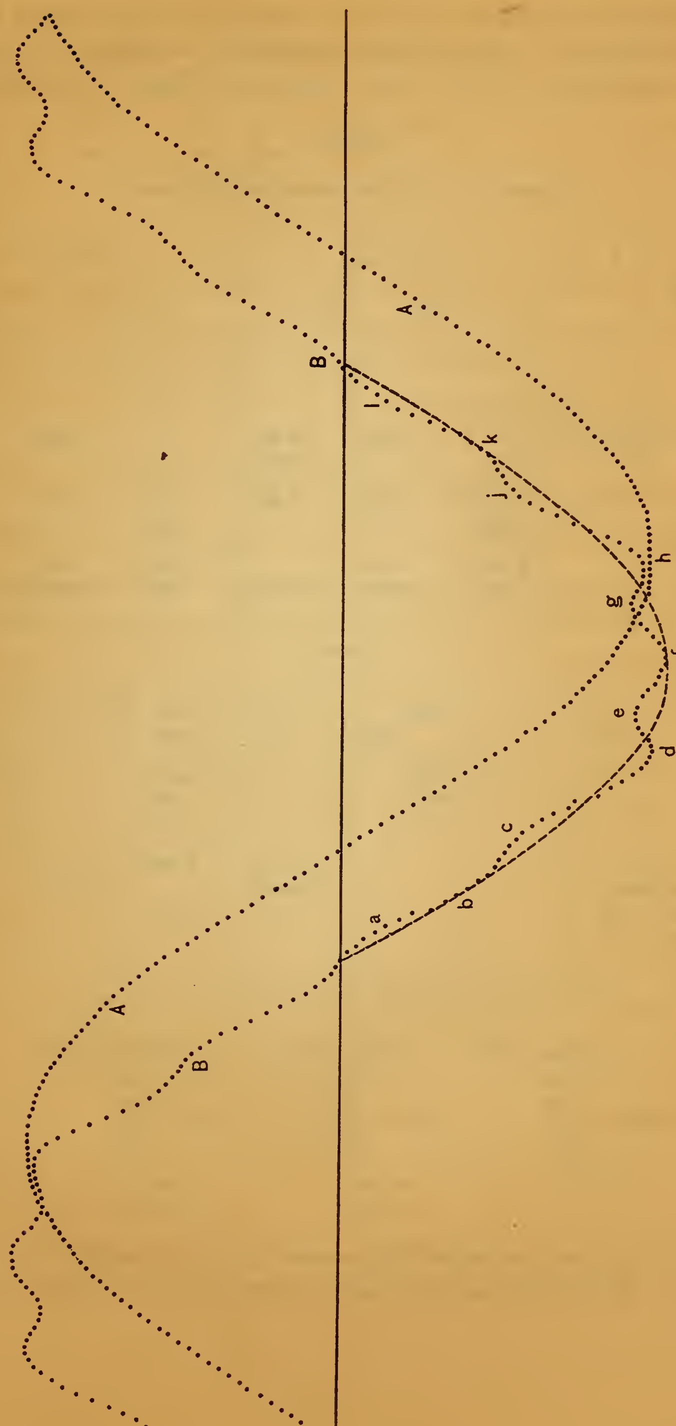

Fig. 1.—*Curve A, current in a highly inductive circuit. Curve B, electromotive force, curve producing the current, curve A*

In the second part of the table is shown the calculation of the check ordinates. The greatest deviation between any fundamental ordinate, and its value calculated from the coefficients

TABLE 5

Analysis of a Curve from Six Ordinates

	Measured ordinates	*s.*	*d.*	*Sine terms* 1st and 5th		*Sine terms* 3d	*Cosine terms* 1st and 5th		*Cosine terms* 3d
				+ −	+ −		+ −	+ −	
0	−1.3		−1.3						
1	15.1 17.3	32.4	−2.2	16.200			0.425		
2	28.0 28.85	56.85	−0.85		49.233	32.400		1.905	−1.300
3	32.35	32.35		32.350		32.350	1.300		0.850
			Sums	48.550	49.233	0.050	1.725	1.905	−0.450
			Se	49.233			1.905		
				97.783	−0.683		−3.630	0.180	
			$A_1=$	32.594	$A_5=-0.228$	$A_3=0.017$	$B_1=-1.210$	$B_5=0.060$	$B_3=-0.150$

CHECK

B_1+B_5	−1.150	A_1+A_5	32.366	(A_1-A_5)	32.822	(B_1-B_5)	−1.270
B_3	−0.150	$-A_3$	−0.017	×2	65.644	×2	−2.540
Sum	−1.300	*Sum*	32.349		56.291		−2.165
y_0	−1.3	y_3	32.35		.554		.035
					.003		
$A_1+A_5=32.366$				$2(A_1-A_5)$	56.848	$2(B_1-B_5)$	−2.200
$A_1-A_5=32.822$				× sin 60°		× sin 60°	
$B_1+B_5=-1.150$				s_2	56.85	d_1	−2.2
$B_1-B_5=-1.270$							

CALCULATION OF THE AMPLITUDES AND PHASES

$$C_1=\sqrt{\overline{32.594}^2+\overline{1.210}^2}=32.616 \qquad \tan\theta_1=\frac{1.210}{32.594}=0.0371 \qquad \theta_1=2^\circ\ 8'$$

$$C_3=\sqrt{\overline{0.017}^2+\overline{0.150}^2}=0.151 \qquad \tan\theta_3=\frac{0.150}{0.017}=8.8 \qquad \theta_3=83^\circ.5 \qquad \phi_3=27^\circ.8$$

$$C_5=\sqrt{\overline{0.228}^2+\overline{0.060}^2}=0.236 \qquad \tan\theta_5=\frac{-0.060}{-0.228}=0.263 \qquad \theta_5=194^\circ.7 \qquad \phi_5=38^\circ.9$$

A_m and B_m is .002, a concordance which may be regarded as satisfactory. The entire work has been carried out to a greater number

	Measured ordinates		
0	0.3		
1	8.5	8.7	17
2	14.3	18.4	32
3	20.6	26.0	46
4	26.15	30.7	56
5	29.8	32.9	62
6	32.25		32

σ_1	σ_2	
17.2	32.7	
46.6	32.25	
63.8	0.45	
62.7		
1.1		
δ_1	δ_2	
−0.2	0.3	
5.4	4.55	
3.1		
8.3	4.85	

	+	
B_1+B_{11}	.	1.
B_3+B_9	1.616	
B_5+B_7	0.525	
	2.141	1.
	1.842	
y_0	0.299	
Actual value	0.3	

TABLE 6

Example of Analysis of Curve from Twelve Measured Ordinates

	Measured ordinates		s	d	Sine terms											
					1st and 11th				3d and 9th				5th and 7th			
					+	−	+	−	+	−	+	−	+	−	+	−
0	0.3			0.3												
1	8.5	8.7	17.2	−0.2	4.452											
2	14.3	18.4	32.7	−4.1			16.350						16.228			
3	20.6	26.0	46.6	−5.4	32.951				0.778						16.350	
4	26.15	30.7	56.85	−4.55			49.233							32.951		
5	29.8	32.9	62.7	−3.1	60.564								16.614			49.233
6	32.25		32.25				32.250				0.450				32.250	
				Sums	97.967		97.833		0.778		0.450		32.842	32.951	48.600	49.233
			S_o		97.967				0.778				−0.109		−0.633	
			S_e		97.833				0.450				−0.633			
						195.800		0.134		1.228		0.328		−0.742		0.524
			Amplitudes			A_1=32.633		A_{11}=0.022		A_3=0.205		A_9=0.055		A_5=0.124		A_7=0.087

			Cosine terms											
			1st and 11th				3d and 9th				5th and 7th			
σ_1	σ_2	0	+	−	+	−	+	−	+	−	+	−	+	−
17.2	32.7	1				0.802								0.052
46.6	32.25	2		2.275								2.275		
63.8	0.45	3				3.818			5.869				3.818	
62.7		4		3.551							3.551			
1.1		5				0.193								2.994
		6	0.300				4.850				0.300			
δ_1	δ_2	*Sums*	0.300	5.826		4.813	4.850		5.869		3.851	2.275	3.818	3.046
		D_o	−5.526				4.850				1.576			
−0.2	0.3	D_e	−4.813				5.869				0.772			
5.4	4.55		−10.339			−0.713	10.719			−1.019	2.348			0.804
3.1														
8.3	4.85	*Amplitudes*		B_1=−1.723		B_{11}=−0.119		B_3=1.786		B_9=−0.170		B_5=0.391		B_7=0.134

CHECK.

	+	−		+	−		+	−		+	−
B_1+B_{11}	.	1.842	A_1-A_{11}	32.611		A_1+A_{11}	32.655		B_1-B_{11}		1.604
B_3+B_9	1.616		$-(A_3-A_9)$		0.150	A_3+A_9	0.260		$-(B_3-B_9)$		1.956
B_5+B_7	0.525		A_5-A_7.		0.211	$-(A_5+A_7)$	0.037		$-(B_5-B_7)$		0.257
	2.141	1.842		32.611	0.361	*Sum*	32.952		*Sum*		3.817
	1.842			0.361		×2	65.944		×2		7.634
y_0	0.299		y_6	32.250		× *sin* 45°	46.601 = s_3		× *sin* 45°		−5.398 = d_3
Actual value	0.3		*Actual value*	32.25		*Actual value*	46.6		*Actual value*		−5.4

CALCULATION OF THE AMPLITUDES AND PHASES.

$C_1=\sqrt{\overline{32.633}^2+\overline{1.723}^2}=32.678$ $\quad tan\ \theta_1=\frac{1.723}{32.633}$ $\quad \theta=3°2'$

$C_3=\sqrt{\overline{0.205}^2+\overline{1.786}^2}=1.798$ $\quad tan\ \theta_3=\frac{-1.786}{0.205}$ $\quad \theta_3=276.°6$ $\quad \psi_3=92.°2$

$C_5=\sqrt{\overline{0.124}^2+\overline{0.391}^2}=0.410$ $\quad tan\ \theta_5=\frac{-0.391}{-0.124}$ $\quad \theta_5=252.°4$ $\quad \psi_5=50.°5$

$C_7=\sqrt{\overline{0.087}^2+\overline{0.134}^2}=0.160$ $\quad tan\ \theta_7=\frac{-0.134}{0.087}$ $\quad \theta_7=303.°0$ $\quad \psi_7=43.°3$

$C_9=\sqrt{\overline{0.055}^2+\overline{0.170}^2}=0.179$ $\quad tan\ \theta_9=\frac{0.170}{0.055}$ $\quad \theta_9=72.°1$ $\quad \psi_9=8.°0$

$C_{11}=\sqrt{\overline{0.119}^2+\overline{0.022}^2}=0.121$ $\quad tan\ \theta_{11}=\frac{0.119}{0.022}$ $\quad \theta_{11}=79.°55$ $\quad \psi_{11}=7.°2$

of significant figures than the accuracy of the curve and the errors of reading the ordinates warrant, since it was desired to illustrate the work using the full number of places given in the multiplication table. If a smaller number of significant figures be retained, the labor is correspondingly reduced.

In the last part of the table are given the values of the amplitudes and phase differences calculated from equations (13). For the calculation of the amplitudes a table of squares and square roots will be found a material aid; the calculation of the phase differences is simplified by the use of a slide rule and a table of natural tangents.

The given curve is represented by the equation

$$I = 32.616 \sin (pt - 2^\circ\ 8') + 0.151 \sin 3(pt - 27^\circ.8) \\ + 0.236 \sin 5(pt - 38^\circ.9) \text{ division}$$

in terms of the arbitrary scale used, or since 20 divisions = 0.5 amperes.

$$I = 0.8154 \sin (pt - 2^\circ\ 8') + 0.00378 \sin 3(pt - 27^\circ.8) \\ + 0.00590 \sin 5(pt - 38^\circ.9) \text{ amperes.}$$

With the origin chosen, the phase of the ordinate y_0 lies $2^\circ\ 8'$ to the left of the zero point of the fundamental. The equation referred to the latter is accordingly

$$I = 0.8154 \sin pt + 0.00378 \sin 3(pt - 25^\circ.7) \\ + 0.00590 \sin 5(pt - 36^\circ.8) \text{ amperes.}$$

EXAMPLE 2. ANALYSIS OF A CURVE BY THE TWELVE-POINT SCHEDULE

In Table 6 is given the complete analysis of one of the emf waves of a small two-phase generator, curve B, Fig. 2. Although the curve is noticeably different from a pure sine wave, no irregularities of very high period are apparent, and the 12-point schedule was used.

The equation of the curve found is

$$E = 32.678 \sin (pt - 3^\circ 2') + 1.798 \sin 3(pt - 92^\circ.2) \\ + 0.410 \sin 5(pt - 50^\circ.5) + 0.160 \sin 7(pt - 43^\circ.3) \\ + 0.179 \sin 9(pt - 8^\circ.0) + 0.121 \sin 11(pt - 7^\circ.2) \\ \text{divisions.}$$

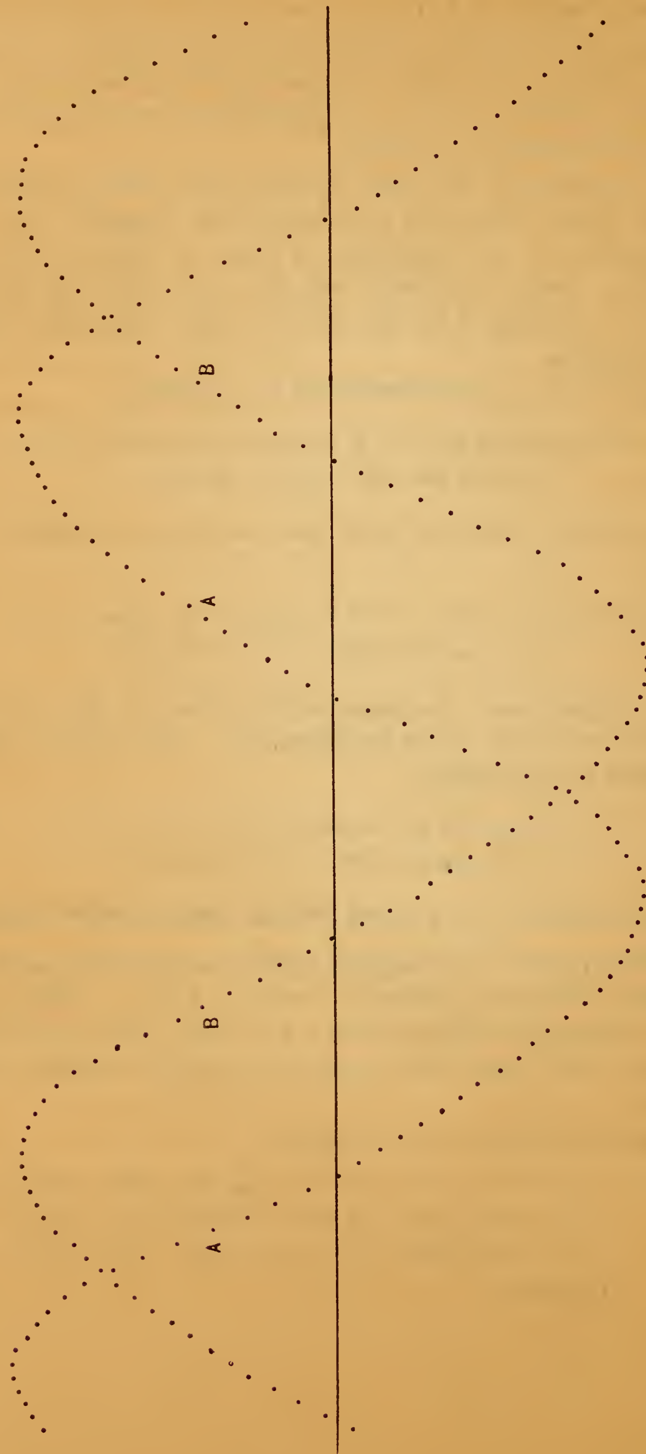

Fig. 2.—Curves of electromotive force of small two-phase machine

EXAMPLE 3. ANALYSIS OF A CURVE BY THE EIGHTEEN-POINT SCHEDULE

The curve A, Fig. 3, is that of the emf. wave of a small generator having 14 slots per pair of poles. This would lead one to expect appreciable harmonics of the thirteenth and fifteenth orders. That this was actually the case is evidenced by the ripples of high frequency on the side of the curve. The curve was, accordingly, analyzed using the 18-point schedule. The complete calculation is shown in Table 7. Although the amount of work is greater than in the two preceding examples, nothing new presents itself.

The results of the analysis show that the amplitudes of the thirteenth and fifteenth harmonics have values as great as several per cent of the fundamental, and are exceeded by the amplitudes of no other harmonics with the exception of the fifth.

The complete equation found is

$$
\begin{gathered}
E = 33.217 \sin (pt - 1^\circ\ 36') + 0.970 \sin 3(pt - 2^\circ.1) \\
+ 1.702 \sin 5(pt - 41^\circ 8) \\
+ 0.218 \sin 7(pt - 38^\circ 3) + 0.355 \sin 9(pt - 2^\circ 3) \\
+ 0.205 \sin 11(pt - 19^\circ 3) \\
+ 1.251 \sin 13(pt - 3^\circ 8) + 1.528 \sin 15(pt - 15^\circ 7) \\
+ 0.159 \sin 17(pt - 3^\circ 0)
\end{gathered}
$$

which shows that, for this curve, an analysis using only 6 points or 12 points would be insufficient.

EXAMPLE 4. CHOICE OF SCHEDULE

In order to illustrate what has been said with respect to the choice of schedule to be used in any given instance, the analysis of each of the three curves in the preceding examples has been carried through with each of the three schedules given here, the results of these analyses being set forth in Table 8.

From this table we see that, for the curve in Example 1, the analysis from 6 points is sufficient for all purposes, the amplitudes of the higher harmonics, found by carrying out the analysis to include more points, are of the order of magnitude of the errors in reading the ordinates, and are not to be regarded as of practical significance. Errors in the curve, due to slight progressive variations in the voltage generated by the machine, where not averaged

Fig. 3.—*Curve A, curve of electromotive force impressed on primary of transformer. Curve B, magnetizing current. Curve C, power used in hysteresis and eddy currents. Curve D, wattless component of the power*

D_o	
D_e	
Amplitudes	B

	+
B_1+B_{17}	
B_3+B_{15}	1.156
B_5+B_{13}	
B_7+B_{11}	0.326
B_9	
Sums	1.482
$y_0=$	0.101
Actual value	0.1

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TABLE 7

Example of Analysis of a Curve from Eighteen Measured Ordinates

	Measured ordinates		s	d	Sine terms: 1st and 17th				3d and 15th				5th and 13th				7th and 11th				9th
					+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	
0	0.1			0.1																	
1	4.6	6.3	10.9	−1.7	1.893									10.732				9.235			
2	10.2	11.2	21.4	−1.0			7.319									16.075				20.145	
3	13.9	18.4	32.3	−4.5	16.150				1.150				16.150					16.150			
4	22.25	24.75	47.0	−2.5			30.211								37.859				13.755		
5	26.8	26.4	53.2	0.4	40.753								8.349				47.341				
6	29.35	32.35	61.7	−3.0			53.433				8.227					53.433			53.433		
7	33.4	28.4	61.8	5.0	58.074									49.992			10.243				
8	28.1	30.8	58.9	−2.7			58.004								21.075					46.286	97.100
9	33.0		33.0		33.000					0.700			33.000					33.000			94.100
Sums					149.870		148.967		1.150	0.700	8.227		57.499	60.724	58.934	69.508	57.584	58.388	67.188	66.431	
S_o					149.870				0.450					−3.225				−0.804			
S_e					148.967				8.227					−10.574				0.757			
						298.837		0.903		8.677		−7.777		−13.799		7.349		−0.047		−1.561	3.000
Amplitudes						A_1 = 33.204		A_{17} = 0.100		A_3 = 0.964		A_{15} = −0.864		A_5 = −1.533		A_{13} = 0.817		A_7 = −0.005		A_{11} = −0.174	A_9 = 0.333

	Cosine terms: 1st and 17th				3d and 15th				5th and 13th				7th and 11th				9th
0	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−	
1		0.469							0.174					0.434			
2			1.710									0.137				0.581	
3		1.500			2.100					1.500				1.500			
4			0.257									1.093				3.214	
5		1.915								2.068			0.766				
6				3.897				6.149			3.897				3.897		
7		0.940							2.349				2.537				
8				1.674							0.394				0.394		3.100
9	0.100				3.100				0.100				0.100				−4.200
Sums	0.100	4.824	1.967	5.571	5.200			6.149	2.623	3.568	8.821	1.230	3.403	1.934	4.291	3.795	
D_o		−4.724			5.200					−0.945			1.469				
D_e		−3.604			−6.149					7.591			0.496				
		−8.328		−1.120		−0.949		11.349		6.646		−8.536		1.965		0.973	−1.100
Amplitudes		B_1 = −0.925		B_{17} = −0.124		B_3 = −0.105		B_{15} = 1.261		B_5 = 0.738		B_{13} = −0.948		B_7 = 0.218		B_{11} = 0.108	B_9 = −0.122

CHECK.

	+	−		+	−		+	−		+
B_1+B_{17}		1.049	A_1+A_{17}	33.304		B_1-B_{17}		0.801	A_1-A_{17}	33.104
B_3+B_{15}	1.156		$-(A_3+A_{15})$		0.100	$-(B_5-B_{13})$		1.686	$-(A_5-A_{13})$	2.350
B_5+B_{13}		0.210	A_5+A_{13}		0.716	$-(B_7-B_{11})$		0.110	(A_7-A_{11})	0.169
B_7+B_{11}	0.326		$-(A_7+A_{11})$	0.179		*Sums*		2.597	*Sum*	35.623
B_9		0.122	A_9	0.333						
Sums	1.482	1.381	*Sums*	33.816	0.816	$\times 2$		5.194	$\times 2$	71.246
y_0	0.101		y_9	33.000		$\times \sin 60°$		$-4.498 = d_3$	$\times \sin 60°$	$61.700 = s_6$
Actual value	0.1		*Actual value*	33.0		*Actual value*		−4.5	*Actual value*	61.7

CALCULATION OF THE AMPLITUDES AND PHASES.

$C_1 = \sqrt{33.204^2+0.925^2} = 33.217$	$\tan\theta_1 = \frac{0.925}{33.204}$	$\theta_1 = 1°36'$	
$C_3 = \sqrt{0.964^2+0.105^2} = 0.970$	$\tan\theta_3 = \frac{0.105}{0.964}$	$\theta_3 = 6°.3$	$\psi_3 = 2°.1$
$C_5 = \sqrt{1.533^2+0.738^2} = 1.702$	$\tan\theta_5 = \frac{-0.738}{-1.533}$	$\theta_5 = 208°.8$	$\psi_5 = 41°.8$
$C_7 = \sqrt{0.005^2+0.218^2} = 0.218$	$\tan\theta_7 = \frac{0.218}{-0.005}$	$\theta_7 = 268°.1$	$\psi_7 = 38°.3$
$C_9 = \sqrt{0.333^2+0.122^2} = 0.355$	$\tan\theta_9 = \frac{0.122}{0.333}$	$\theta_9 = 20°.8$	$\psi_9 = 2°.3$
$C_{11} = \sqrt{0.174^2+0.108^2} = 0.205$	$\tan\theta_{11} = \frac{0.108}{-0.174}$	$\theta_{11} = 211°.8$	$\psi_{11} = 19°.3$
$C_{13} = \sqrt{0.817^2+0.945^2} = 1.251$	$\tan\theta_{13} = \frac{0.948}{0.817}$	$\theta_{13} = 49°.3$	$\psi_{13} = 3°.8$
$C_{15} = \sqrt{0.864^2+1.261^2} = 1.528$	$\tan\theta_{15} = \frac{-1.261}{-0.864}$	$\theta_{15} = 235°.6$	$\psi_{15} = 15°.7$
$C_{17} = \sqrt{0.100^2+0.124^2} = 0.159$	$\tan\theta_{17} = \frac{0.124}{0.100}$	$\theta_{17} = 51°.1$	$\psi_{17} = 3°.0$

TABLE 8

Comparison of the Results of Analyses of the Same Curves Using Different Numbers of Ordinates

Curve of Example 1

	6 points	12 points	18 points		6 points	12 points	18 points
C_1	32.616	32.641	32.648	ψ_1	2°8′	1°55′	1°52′
C_3	0.151	0.103	0.096	ψ_3	27°.8	26°.5	35°.5
C_5	0.236	0.291	0.288	ψ_5	38°.9	36°.2	37°.0
C_7		0.085	0.082	ψ_7		30°.8	29°.0
C_9		0.073	0.075	ψ_9		12°.8	15°.4
C_{11}		0.108	0.124	ψ_{11}		61°.8	5°.5
C_{13}			0.019	ψ_{13}			6°.5
C_{15}			0.032	ψ_{15}			3°.3
C_{17}			0.011	ψ_{17}			11°.2

Curve of Example 2

	6 points	12 points	18 points		6 points	12 points	18 points
C_1	32.663	32.678	32.773	ψ_1	3°14′	3°2′	3°22′
C_3	1.623	1.798	1.775	ψ_3	91°.8	92°.2	93°.1
C_5	0.566	0.410	0.507	ψ_5	49°.6	50°.5	51°.8
C_7		0.160	0.183	ψ_7		43°.3	44°.3
C_9		0.179	0.221	ψ_9		8°.0	6°.8
C_{11}		0.121	0.180	ψ_{11}		7°.2	0°.8
C_{13}			0.145	ψ_{13}			24°.4
C_{15}			0.064	ψ_{15}			16°.4
C_{17}			0.113	ψ_{17}			5°.0

Curve of Example 3

	6 points	12 points	18 points		6 points	12 points	18 points
C_1	34.240	33.194	33.217	ψ_1	2°50′	1°42′	1°36′
C_3	1.059	1.051	0.970	ψ_3	85°.8	−0°.8	2°.1
C_5	1.653	1.680	1.702	ψ_5	42°.3	40°.2	41°.7
C_7		0.268	0.218	ψ_7		34°.1	38°.3
C_9		1.619	0.355	ψ_9		−4°.2	2°.2
C_{11}		1.279	0.205	ψ_{11}		13°.0	19°.3
C_{13}			1.251	ψ_{13}			3°.8
C_{15}			1.528	ψ_{15}			15°.7
C_{17}			0.159	ψ_{17}			3°.0

out by taking two half waves, affect all the analyses in nearly the same way, especially since the fundamental ordinates used are in

part common to all three analyses. Sudden fluctuations of short duration tend to give spurious values for the higher harmonics. Such can, however, be detected by repeating the analyses using other sets of ordinates, as shown in example 5 below.

In the case of the curve of Example 2, the analysis from six points gives a very satisfactory degree of accuracy for the third and fifth harmonics, but the amplitudes of the seventh, ninth and eleventh harmonics, found by using more points, are greater than in some cases could be neglected.

In the case of the curve in Example 3, neither the 6-point nor the 12-point analysis is sufficient. In the former the amplitude of the fundamental and the phase of the third harmonic are considerably in error; in the latter, neglecting the thirteenth and fifteenth harmonics, gives erroneous values for the ninth and eleventh harmonics.

The inadequacy of the 6-point and 12-point analyses, in Example 3, may be further shown by calculating the values of the ordinates of points not used in the analyses for comparison with the actual values taken from the curve. Thus, from the schedules of Table 4, the values given in the following table were calculated. The

TABLE 9

Calculated Values of Ordinates of Curve in Example 3 for Points not Used in the Analyses

18 points				6 points				12 points			
Phase	Ordinates		Differences	Phase	Ordinates		Differences	Phase	Ordinates		Differences
	Observed	Calculated			Observed	Calculated			Observed	Calculated	
15°	7.55	7.49	+0.06	15°	7.55	7.02	+0.53	10°	4.6	4.75	−0.15
45°	25.85	25.83	+0.02	45°	25.85	23.99	+1.86	20°	10.2	9.61	+0.59
75°	29.95	30.64	−0.69	75°	29.95	31.54	−1.59	40°	22.25	22.43	−0.18
105°	28.0	27.68	+0.32	105°	28.0	30.45	−2.45	50°	26.8	27.08	−0.28
135°	25.35	25.10	+0.25	135°	25.35	26.50	−1.15	70°	33.4	30.91	+2.49
165°	9.1	9.08	+0.02	165°	9.1	8.57	+0.53	80°	28.1	30.86	−2.76
								100°	30.8	30.80	0
								110°	28.4	30.12	−1.72
								130°	26.4	28.37	−1.97
								140°	24.75	22.63	+2.12
								160°	11.2	11.97	−0.77
								170°	6.3	5.73	+0.57

phases of the various points refer to the same origin as was used in the analyses.

With the exception of one point, the calculated and actual values of these ordinates, using the 18-point analysis, are in satisfactory agreement; the deviations in the case of the 6-point and 12-point analysis are, however, very appreciable, and the 12-point analysis has very little advantage in this respect over the 6-point.

The deviations were also calculated for the curve in Example 2, but were not greater than about 0.5 division in the case of all three analyses, the agreement being no better when 18 points were used than with only 6. This procedure does not, therefore, give much information, except when the neglected harmonics are of relatively large importance. It will generally be found that a decision regarding the sufficiency of any given analysis will be more rapidly and certainly reached by carrying through another analysis, using the same schedule, but with an independent set of ordinates.

EXAMPLE 5. ACCURACY OF RESULTS

To throw light on the question as to what precision has been reached in the preceding examples, analyses were made using other sets of ordinates than those on which these examples were based. The results are given in Table 10. The different sets of ordinates are distinguished by reference to the relative phases of y_o referred to the zero points of the preceding analyses.

TABLE 10

Comparison of Results of Analyses of Same Curves Using Different Sets of Ordinates

CURVE OF EXAMPLE 1

6-point analyses											
	0°	2°	4°	6°	8°		0°	2°	4°	6°	8°
C_1	32.616	32.623	32.607	32.653	32.622	ψ_1	2°8′	2°4′	2°3′	1°52′	1°52′
C_3	0.151	0.118	0.186	0.061	0.151	ψ_3	27.8	29.2	30.6	19.6	40.2
C_5	0.236	0.260	0.218	0.209	0.270	ψ_5	38.9	36.1	36°.0	34.7	30°.3

TABLE 10—Continued

12-point analyses

	0°	2°	4°	6°	8°		0°	2°	4°	6°	8°
C_1	32.641	....	....	32.605	....	ψ_1	1°55′	....	....	1°46′	....
C_3	0.103	....	....	0.028	....	ψ_3	26°.5	....	....	33.3	....
C_5	0.291	....	....	0.317	....	ψ_5	36°.2	....	....	36.1	....
C_7	0.085	....	....	0.122	....	ψ_7	30.8	....	....	27.9	....
C_9	0.073	....	....	0.074	....	ψ_9	12.8	....	....	22.5	....
C_{11}	0.108	....	....	0.088	....	ψ_{11}	5.6	....	....	12.7	....

CURVE OF EXAMPLE 2

12-point analyses						18-point analyses					
	0°	4°		0°	4°		0°	4°		0°	4°
C_1	32.678	32.658	ψ_1	3°2′	2°59′	C_1	32.733	32.690	ψ_1	3°22′	2°51′
C_3	1.798	1.839	ψ_3	92°.2	92°.4	C_3	1.775	1.814	ψ_3	93°.1	91°.6
C_5	0.410	0.346	ψ_5	50.5	48.5	C_5	0.507	0.401	ψ_5	51°.8	51.6
C_7	0.160	0.053	ψ_7	43.3	39.9	C_7	0.183	0.061	ψ_7	44.3	44.6
C_9	0.179	0.137	ψ_9	8.0	1.2	C_9	0.221	0.077	ψ_9	6.8	2.1
C_{11}	0.121	0.179	ψ_{11}	7.2	2.3	C_{11}	0.180	0.104	ψ_{11}	0.8	2.6
						C_{13}	0.145	0.006	ψ_{13}	24.4	28.2
						C_{15}	0.064	0.020	ψ_{15}	16.4	20.3
						C_{17}	0.113	0.047	ψ_{17}	5.0	8.0

CURVE OF EXAMPLE 3

12-point analyses						18-point analyses					
	0°	6°		0°	6°		0°	6°		0°	6°
C_1	33.194	33.187	ψ_1	1°42′	1°28′	C_1	33.217	33.202	ψ_1	1°36′	1°36′
C_3	1.051	0.972	ψ_3	−0°.8	5°.9	C_3	0.970	1.025	ψ_3	2°.1	3°.6
C_5	1.680	1.765	ψ_5	40.2	40.8	C_5	1.702	1.692	ψ_5	41.7	40.4
C_7	0.268	0.339	ψ_7	34.1	36.8	C_7	0.218	0.346	ψ_7	38.3	37.3
C_9	1.619	1.728	ψ_9	35.8	9.0	C_9	0.355	0.450	ψ_9	2.2	5.0
C_{11}	1.279	1.801	ψ_{11}	13.0	24.6	C_{11}	0.205	0.126	ψ_{11}	19.3	20.5
						C_{13}	1.251	1.275	ψ_{13}	3.8	3.5
						C_{15}	1.528	1.507	ψ_{15}	15.7	15.3
						C_{17}	0.159	0.064	ψ_{17}	3.0	17.8

An examination of the table shows that analyses depending on different sets of ordinates give the amplitudes of harmonics, whose values are as great as about 0.2 division, with a very fair degree of accuracy, both as regards amplitude and phase. Different deter-

minations of harmonics of amplitudes as large as one division do not differ by more than a few per cent in amplitude nor more than a degree or two in the value of ψ. (The accuracy of reading of the fundamental ordinates is not greater than about 0.1 division.) The preceding assumes, however, that a sufficient number of harmonics have been included in the analysis. In the case of the curve of Example 3 as analyzed by the 12-point schedule, for instance, the effect of the thirteenth and fifteenth harmonics is distributed between the three highest harmonics included, and their values are different, both in amplitude and phase in analyses using different sets of ordinates.

In answer to the question as to how small harmonics in the above examples have been proved to exist, there seems to be reason to believe that, in some cases, the analysis has located with some degree of accuracy, harmonics of amplitude as small as 0.1 division which, considering that this is about the order of errors in reading the ordinates, points to a very satisfactory performance of the curve tracer.

EXAMPLE 6. PREDICTION OF PRESENCE OF HARMONICS FROM THE APPEARANCE OF THE CURVE

It is of interest to consider, in the case of distorted waves, how far one may predict from the appearance of the curves what important harmonics are present. Provided only one harmonic is present to a notable extent, its presence is manifested by a waviness of the curve, and the number of the crests and troughs per half wave of the fundamental will be equal to the order of the harmonic. When, however, more than one harmonic is important, it will not, in general, be possible to locate them all by counting the crests and troughs superposed on the fundamental.

As an example, we may take the curve of Example 3 (Curve A, Fig. 3). Fifteen maxima and minima (of very different amplitudes it is true) may be counted in each half wave of the curve. However, analysis has already shown that not only is the fifteenth harmonic prominent, but also the thirteenth, third, and fifth, to about an equal extent, while the amplitudes of the seventh, ninth, and eleventh are by no means negligible. It is interesting to study how these various harmonics combine in their effects to produce

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the observed distortion of the wave. In Fig. 3 is plotted in dashed lines the fundamental wave given by the analysis. In Fig. 4 the curve A shows the deviations of the distorted wave from the fundamental given by the analysis, as a function of the phase, referred to the arbitrary origin of the analysis. This origin lies about 1°.6 to the left of the zero of the fundamental. The dotted curves in Fig. 4 show graphically the amplitudes and phases of the third, fifth, thirteenth, and fifteenth harmonics given by the analysis. The plotting of these waves is an easy matter, since for those of the higher frequencies, little needs to be done except to indicate the position of the zero points and maxima and minima, the rest of the curve being drawn in free hand. The curve B defined by the little circles is the resultant of the waves of the third, fifth, thirteenth, and fifteenth harmonics. The period of the ripples lies somewhere between that of the thirteenth and fifteenth harmonics. The small deviations of the actual curve from this resultant curve may be shown to be

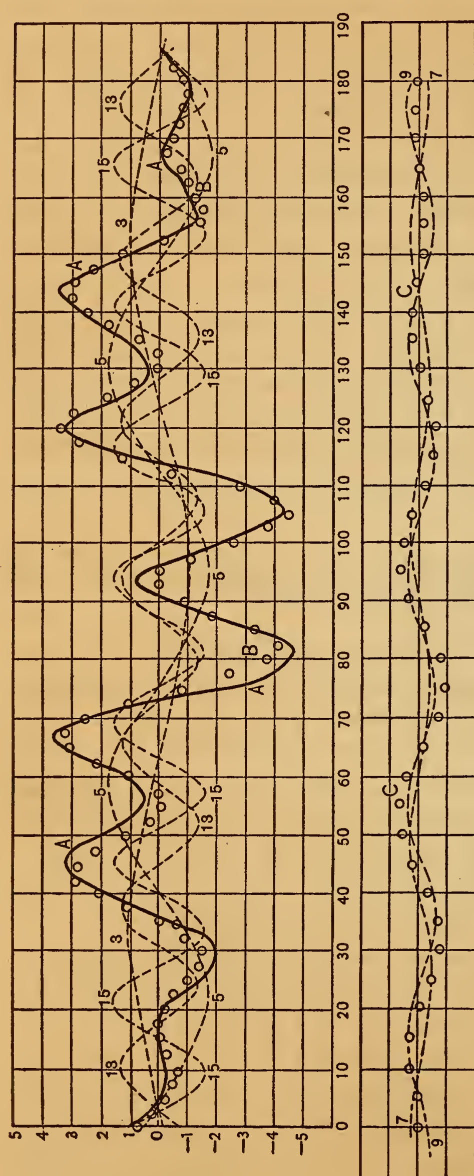

Fig. 4.—Curves showing distorting effects of the various harmonics in electromotive force wave, curve A, Fig. 3.

reduced practically to zero when the remaining harmonics are taken into consideration. These have been omitted for the sake of clearness, but the curves of the seventh and ninth harmonics, together with their resultant, C, are drawn in the lower part of the figure. If the ordinates of curve C are added to those of curve B the resulting curve agrees very closely with the actual curve A.

This example shows, therefore, that in many cases it is not possible to tell from the appearance of the curve what harmonics are present. In the present instance, from the number of ripples the presence of the fifteenth harmonic only could be predicted with certainty, while the almost equally important thirteenth and the still greater fifth harmonics would not be included. Naturally, however, the varying height of the different troughs and crests would lead one to suspect more than one harmonic.

EXAMPLE 7. BADLY DISTORTED CURVES—INFLUENCE OF CAPACITY AND INDUCTANCE

In the curves considered in the previous examples, the curves, although by no means closely approaching pure sine curves in character, are very far from showing the irregularities in wave form which often occur in circuits containing capacity. The curve B of Fig. 5 shows the current through a capacity of nine microfarads. The emf impressed on the condensers (and a resistance of 0.7 ohm joined in series for drawing the curve) is curve A of the same figure.

Analysis of the current curve by means of the 18-point schedule gives the following values of the amplitudes and phases of the components:

		Values reduced to emf.
$C_1 = 22.84$	$\psi_1 = -81.7^\circ$	
$C_3 = 2.28$	$\psi_3 = +38.3$	1.17
$C_5 = 7.22$	$\psi_5 = 30.5$	2.21
$C_7 = 11.48$	$\psi_7 = 8.0$	2.53
$C_9 = 1.51$	$\psi_9 = 10.8$	0.26
$C_{11} = 7.17$	$\psi_{11} = 27.9$	1.00
$C_{13} = 0.28$	$\psi_{13} = 13.0$	0.03
$C_{15} = 0.85$	$\psi_{15} = 16.8$	0.09
$C_{17} = 0.66$	$\psi_{17} = 1.3$	0.06

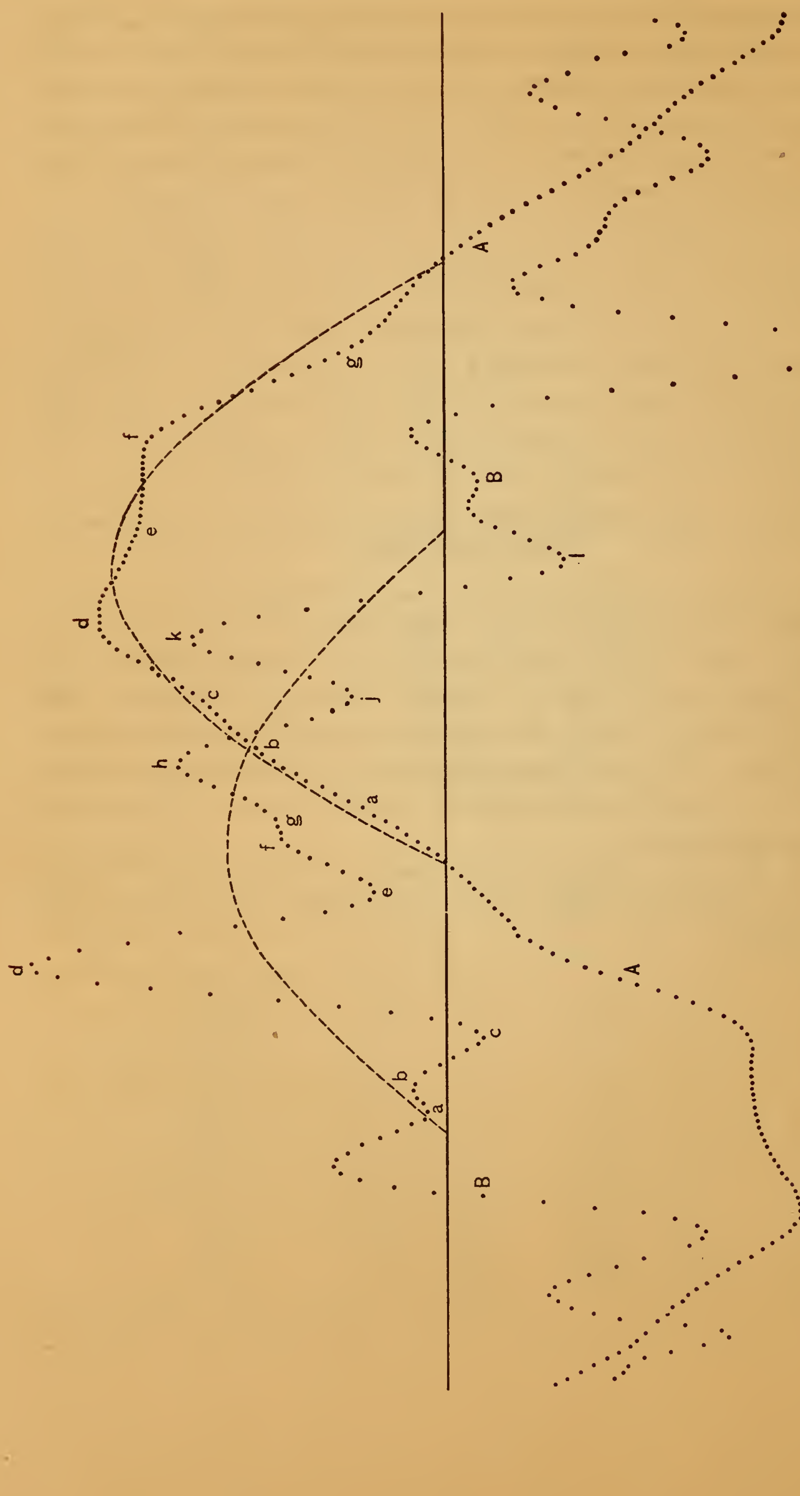

Fig. 5.—*Curve A, electromotive force wave impressed. Curve B, current in condenser circuit*

On calculating the deviations of this curve from the actual, using the scheme of calculation in Table 4 for this purpose, it was found that the departure is on the average not greater than about one division, an agreement which is to be regarded as satisfactory when the extreme steepness of the curve in some parts of its course is considered. Analysis of the emf wave gave

$C_1 = 35.01$	$\psi_1 = -2.1^\circ$
$C_3 = 0.99$	$\psi_3 = +58.2$
$C_5 = 1.90$	$\psi_5 = 38.3$
$C_7 = 1.93$	$\psi_7 = 12.2$
$C_9 = 0.09$	$\psi_9 = 11.2$
$C_{11} = 0.45$	$\psi_{11} = 26.4$
$C_{13} = 0.18$	$\psi_{13} = 13.5$
$C_{15} = 0.08$	$\psi_{15} = 4.0$
$C_{17} = 0.03$	$\psi_{17} = 20.5$

In a circuit containing capacity, but in which the resistance is negligible, the harmonics in the emf. wave are magnified in the current wave in proportion to the order of the harmonic. To test the accuracy of this relation in the case of the curves just given, the third column in the results of the analysis of the current curve given above, the column designated as "values reduced to emf.," was obtained by multiplying each harmonic in the current wave by the ratio of the fundamentals $\left(\frac{35.01}{22.84}\right)$ and dividing by the order of the harmonic in question. The general qualitative agreement of these calculated values with the harmonics of the emf wave is perhaps as good as could be expected when it is remembered that in addition to the 0.7 ohm the rest of the circuit was not of negligible resistance.

The same generator when run on an inductive load gave an emf. curve B, Fig. 1, which yielded on analysis

$$
\begin{array}{ll}
C_1 = 33.96 & \psi_1 = -\ 1.0^{\circ} \\
C_3 = 0.34 & \psi_3 = +60.6 \\
C_5 = 0.93 & \psi_5 = 35.9 \\
C_7 = 0.75 & \psi_7 = 50.8 \\
C_9 = 0.14 & \psi_9 = 1.0 \\
C_{11} = 1.50 & \psi_{11} = 16.5 \\
C_{13} = 0.27 & \psi_{13} = 4.7 \\
C_{15} = 0.08 & \psi_{15} = 20.7 \\
C_{17} = 0.09 & \psi_{17} = 18.0
\end{array}
$$

The armature has 12 slots per cycle.

Not only is the shape of the curve very different from that given on a capacity load, curve A, Fig. 5, but, whereas in the latter case the third, fifth, and seventh harmonics were present to a marked degree, the eleventh harmonic is most prominent on inductive load. This gives a very striking illustration of the effects of armature reactions.

The current in the inductive circuit, curve A, Fig. 1, has already been analyzed and discussed in Example 1. The results show that the effect of inductance is to smooth out the harmonics in the emf. to an extent which is proportional to the order of the harmonic. For a more accurate experimental verification of this principle, the reader is referred to an article in this Bulletin.[8]

TABLE 11

Showing Relative Values of the Harmonics to Form the Crests and Troughs of the Curves in Example 7

CONDENSER CURRENT

Crest or trough	Harmonics				Sum	Observed deviation
	3	5	7	11		
a	1.0	−3.6	3.6	−3.6	− 2.6	− 2.3
b	1.6	−6.2	− 5.0	5.0	− 4.6	− 4.0
c	2.2	−6.0	−10.2	−2.2	−16.2	−15.2
d	0.7	6.5	11.0	6.9	25.1	24.8
e	−1.3	1.7	−10.4	−5.0	−15.0	−14.8
f	−2.2	−6.6	0.0	5.0	− 3.8	− 5.4
g	−2.0	−7.1	9.0	−3.5	− 3.6	− 4.1
h	−0.7	−0.7	5.0	2.7	6.3	6.4
j	1.2	7.1	−11.3	−5.0	− 8.0	− 7.3
k	2.2	−1.0	10.4	6.9	18.5	15.0
l	1.0	−6.4	− 4.8	−4.8	−15.0	−15.5

[8] This Bulletin, I, p. 140; 1904.

TABLE 11—Continued

CONDENSER ELECTROMOTIVE FORCE

Crest or trough	Harmonics			Sum	Observed deviation
	3	5	7		
a	−0.6	−1.4	0.4	−1.6	−2.2
b	−0.9	−0.6	1.1	−0.4	−0.7
c	−0.5	1.0	−1.3	−0.8	−1.5
d	0.7	0.7	1.8	3.2	3.2
e	0.6	−1.4	−1.9	−2.7	−2.2
f	−0.7	1.7	1.7	2.7	3.4
g	−1.0	−1.0	−1.9	−3.9	−4.4

INDUCTIVE ELECTROMOTIVE FORCE

Crest or trough	Harmonics				Sum	Observed deviation
	3	5	7	11		
a	−0.2	−0.5	0.5	−1.5	−1.7	−1.8
b	−0.3	−0.8	0.3	1.3	0.5	0.0
c	−0.4	0.2	−0.7	−1.3	−2.2	−2.4
d	0.0	0.9	0.6	1.4	2.9	2.3
e	0.2	−0.5	−0.3	−1.1	−1.7	−2.1
f	0.4	−0.9	−0.7	1.3	0.1	0.0
g	0.3	−0.5	0.0	−1.2	−1.4	−1.4
h	0.0	0.8	0.5	1.4	2.7	2.3
j	−0.3	0.3	−0.7	−1.4	−2.1	−2.4
k	−0.3	−0.7	0.0	1.4	0.4	0.0
l	−0.1	0.5	−0.7	−1.4	−1.7	−1.6

It is of interest, in the case of each of these curves, to see how the shape of the curve depends on the principal harmonics present. In each case the fundamental component, derived by the analysis, is plotted in the figure as a dashed sine curve. Table 11 shows to what harmonics each trough and crest is due. For purposes of identification, these have been lettered. In the table are given the values of the harmonic components at the points in question, as derived from a plot of the results of the analysis, together with their resultant (algebraic sum). For comparison there are appended the actual measured deviations of the curve from the fundamental given by the analysis (the dashed sine curve). No attempt was made to obtain an accurate comparison, the table being given for purposes of orientation only.

EXAMPLE 8. RESONANCE OF A PARTICULAR HARMONIC

It sometimes happens that the inductance and capacity are of such values as to give resonance for some particular harmonic of the emf. wave. The curve *A*, Fig. 6, is an emf. wave taken from the same machine as in Example 3. The thirteenth and fifteenth harmonics are appreciable, amounting to about 3.8 per cent and 4.6 per cent of the fundamental, respectively. In this particular case it was found that on joining a Siemens dynamometer in the circuit to measure the current through a small capacity, the current indicated by the dynamometer was much larger than was expected from the known approximate value of the capacity. On drawing the current curve, the wave form was found to be that of curve *B*; the higher harmonics are seen at a glance to be very prominent. Analysis gave the following results:

$C_1 = 1.57$	$\psi_1 = -88.1°$
$C_3 = 0.51$	$\psi_3 = -\ 0.1$
$C_5 = 0.23$	$\psi_5 = -\ 7.0$
$C_7 = 1.21$	$\psi_7 = 32.6$
$C_9 = 0.48$	$\psi_9 = 9.8$
$C_{11} = 1.75$	$\psi_{11} = 21.7$
$C_{13} = 1.21$	$\psi_{13} = 14.6$
$C_{15} = 4.94$	$\psi_{15} = 3.1$
$C_{17} = 0.63$	$\psi_{17} = -\ 3.3$

All the harmonics included in the analysis are appreciable, the seventh, eleventh, and thirteenth being of the same order of magnitude as the fundamental, while the fifteenth harmonic has an amplitude of more than three times that of the fundamental, the condition of resonance being most completely realized for this harmonic. In fact, if the components be plotted, it is found that the successive maxima and minima of the curve coincide almost exactly with those of the fifteenth harmonic, although the remaining component waves influence the actual values of these maxima and minima to an extent which varies according as they act in conjunction or opposition.

Fig. 6.—Curve A, electromotive force wave. Curve B, current curve. Resonance of fifteenth harmonic

The phases are referred to an origin taken arbitrarily at the extreme left-hand point of the current wave. The emf wave does not seem to have been drawn in the proper phase relation with the current curve.

In such cases as those in which resonance renders abnormally prominent some harmonic of an order higher than is included in the foregoing schedules of analysis, it is not difficult to derive a scheme of analysis for the single harmonic in question, according to the principles already laid down.

V. FURTHER APPLICATIONS

1. CALCULATION OF THE AVERAGE AND EFFECTIVE VALUES OF ELECTROMOTIVE FORCE AND CURRENT CURVES

It is customary to define the *average* value of an emf or current wave as the average ordinate of one loop of the curve (either positive or negative); since the positive and negative loops are alike the average of the ordinates throughout a complete period is zero. In the case of very distorted waves, as, for instance, that of the condenser current, Fig. 5, it is difficult, if not impossible, to state where is the beginning of a loop. The usual custom of taking as the beginning of the loop the point where the curve passes through zero, evidently becomes inapplicable when the curve is constantly oscillating between positive and negative values. The average value of an emf. or current wave is consequently a quantity which depends on the choice of origin of phase differences, and it is difficult to define it in such a way as to include the customary rather special definition of this quantity. In what follows the average value of an emf. or current wave will be defined as the average ordinate of one-half wave, counting phase differences from the *zero of the fundamental* as origin. This definition does not give quite the same value as that found by taking as origin the point (in the case of only slightly distorted waves) where the curve crosses the axis, but the difference will, in general, be found small.

With this understanding, and writing for the equation of the wave

$$e = E_1 \sin \frac{2\pi}{T} t + E_3 \sin \left[3\left(\frac{2\pi}{T}\right)t - \beta_3\right] + \ . \ . \ .$$
$$+ E_n \sin \left[n\left(\frac{2\pi}{T}\right)t - \beta_n\right] + \ . \ . \ .$$

the average value ϵ is defined by the expression

$$\epsilon=\frac{2}{T}[E_1\int_0^{\frac{T}{2}}\sin\frac{2\pi}{T}t\,dt+E_3\int_0^{\frac{T}{2}}\sin\left(3\frac{2\pi}{T}t-\beta_3\right)dt+\ \dots$$
$$+E_n\int_0^{\frac{T}{2}}\sin\left(n\frac{2\pi}{T}t-\beta_n\right)dt+\ \dots\ ]$$

where T is the period of the fundamental.

We have, therefore, to evaluate the general integral

$$\int_0^{\frac{T}{2}}\sin\left(\frac{2\pi n}{T}t-\beta_n\right)dt$$

which is easily found to be equal to $\frac{T}{\pi n}\cos\beta_n$, so that the final expression for the average value is

$$\epsilon=\frac{2}{\pi}\Sigma\frac{E_n}{n}\cos\beta_n \tag{14}$$

where $$\beta_n=n(\psi_n-\psi_1)$$

and ψ_n has the same meaning as in equation (13).

The *root mean square* or *effective* value is involved in no such ambiguity as the *average* value, being entirely independent of what point is chosen as origin. Writing the equation of the curve in the form

$$e=E_1\sin\left(\frac{2\pi}{T}t-\theta_1\right)+E_3\sin\left(3\frac{2\pi}{T}t-\theta_3\right)+\ \dots$$
$$+E_n\sin\left(n\frac{2\pi}{T}t-\theta_n\right)+\ \dots$$

we have for the square of the instantaneous value

$$e^2=E_1^2\sin^2\left(\frac{2\pi}{T}t-\theta_1\right)+E_3^2\sin^2\left(3\frac{2\pi}{T}t-\theta_3\right)+\ \dots$$
$$+E_n^2\sin^2\left(n\frac{2\pi}{T}t-\theta_n\right)+\ \dots$$
$$+2E_1E_n\sin\left(\frac{2\pi}{T}t-\theta_1\right)\sin\left(n\frac{2\pi}{T}t-\theta_n\right)+\ \dots$$
$$+2E_3E_n\sin\left(3\frac{2\pi}{T}t-\theta_3\right)\sin\left(n\frac{2\pi}{T}t-\theta_n\right)+\ \dots$$

and the square of the effective value will be $\mathbf{E}^2 = \frac{1}{T}\int_0^T e^2 dt.$ The integrals of the form $\int_0^T \sin\left(m\frac{2\pi}{T}t - \theta_m\right)\sin\left(n\frac{2\pi}{T}t - \theta_n\right)dt$ are easily seen to be each separately equal to zero, so that there remain only integrals of the general form $\int_0^T \sin^2\left(n\frac{2\pi}{T}t - \theta_n\right)dt$, which, using the relation $\int \sin^2 x dx = \frac{x}{2} - \frac{\sin x \cos x}{2}$, are found to give merely the term $\frac{T}{2}$ so that we have the simple expression

$$\mathbf{E}^2 = \frac{1}{2}(E_1{}^2 + E_3{}^2 + \ . \ . \ . \quad + E_n{}^2 + \ . \ . \ .) \tag{15}$$

2. CALCULATION OF THE AVERAGE POWER IN A CIRCUIT

The average power in a circuit is found by taking the average of the values of the instantaneous power (ei) through an entire period. This may, of course, be calculated by measuring a sufficient number of ordinates of the emf and current curves, and by forming the products of corresponding values of the ordinates of the two waves. The average power may, however, be much more readily and more accurately calculated from the results of an analysis of the current and emf waves. Writing as usual

$$e = E_1 \sin\left(\frac{2\pi}{T}t - \theta_1\right) + E_3 \sin\left(3\frac{2\pi}{T}t - \theta_3\right) + \ . \ . \ .$$
$$+ E_n \sin\left(n\frac{2\pi}{T}t - \theta_n\right) + \ . \ . \ .$$

$$i = I_1 \sin\left(\frac{2\pi}{T}t - \delta_1\right) + I_3 \sin\left(3\frac{2\pi}{T}t - \delta_3\right) + \ . \ . \ .$$
$$+ I_n \sin\left(n\frac{2\pi}{T}t - \delta_n\right) + \ . \ . \ .$$

the instantaneous power is given by

$$
\begin{aligned}
ei = {} & E_1I_1 \sin\left(\frac{2\pi}{T}t - \theta_1\right) \sin\left(\frac{2\pi}{T}t - \delta_1\right) \\
& + E_3I_3 \sin\left(3\frac{2\pi}{T}t - \theta_3\right) \sin\left(3\frac{2\pi}{T}t - \delta_3\right) + \ . \ . \ . \\
& + E_1I_3 \sin\left(\frac{2\pi}{T}t - \theta_1\right) \sin\left(3\frac{2\pi}{T}t - \delta_3\right) + \ . \ . \ . \ + \\
& E_1I_n \sin\left(\frac{2\pi}{T}t - \theta_1\right) \sin\left(n\frac{2\pi}{T}t - \delta_n\right) + \ . \ . \ . \ . \\
& + E_3I_1 \sin\left(3\frac{2\pi}{T}t - \theta_3\right) \sin\left(\frac{2\pi}{T}t - \delta_1\right) \\
& + E_3I_5 \sin\left(3\frac{2\pi}{T}t - \theta_3\right) \sin\left(5\frac{2\pi}{T}t - \delta_5\right) + \ . \ . \ .
\end{aligned}
$$

and the average power is found by performing the integration of this expression between the limits zero and T, and by then dividing the result by T. The integrals of the form $\int_0^T \sin\left(m\frac{2\pi}{T}t - \theta_m\right) \sin\left(n\frac{2\pi}{T}t - \delta_n\right)dt$ are all separately equal to zero. The remaining integrals are of the form

$$
\int_0^T \sin\left(n\frac{2\pi}{T}t - \theta_n\right) \sin\left(n\frac{2\pi}{T}t - \delta_n\right)dt =
$$

$$
\frac{1}{2}\int_0^T \left[\cos\,(\theta_n - \delta_n) - \cos\left\{2n\frac{2\pi}{T}t - (\theta_n + \delta_n)\right\}\right]dt
$$

which, since $\int_0^T \cos\left[2n\frac{2\pi}{T}t - (\theta_n + \delta_n)\right]dt = \text{zero}$ are seen to give the final expression for the average power

$$
\begin{aligned}
P &= \frac{1}{2}[E_1I_1 \cos\,(\theta_1 - \delta_1) + E_3I_3 \cos\,(\theta_3 - \delta_3) + \ . \ . \ . \ + E_nI_n \cos(\theta_n - \delta_n)] \qquad (16) \\
&= \frac{1}{2}[E_1I_1 \cos\,\varphi_1 + E_3I_3 \cos\,\varphi_3 + \ . \ . \ . \ + E_nI_n \cos\,\varphi_n + \ . \ . \ . \ ]
\end{aligned}
$$

where φ_n is the angle of phase difference between the harmonic emf. of n times the frequency of the fundamental and the current component of the same frequency. This equation expresses the well-known fact that on the average electromotive forces and currents of different frequencies add nothing to the power of a circuit.

The power factor of a circuit, in which the emf and current may have any wave form whatever, is defined as the ratio of the average power to the product of the effective emf. by the effective current. In the case of pure sine waves it is equal to the cosine of the phase difference between the emf. and the current; in the case of distorted waves, the power factor may be designated as the cosine of an angle, which we may call the *effective phase difference*, an angle which has no physical existence, but which is equal to the phase difference which would have to exist between two sine waves having, respectively, the same effective values as the emf. and current, and developing the same average power as the actual emf. and current.

We thus see that if the results of the analysis of the emf. and current curves in a circuit are at hand, it is an easy matter to calculate the average and effective values of the emf and current, the form factor, the average power, the power factor, and the effective phase difference between the emf. and the current. Of these quantities, some may be calculated from the curves, by measuring a sufficient number of ordinates, and forming the averages of their squares and products. The *average* values of emf. and current (and therefore their form factors) can not, however, be accurately found without having analyzed the curves (in the case of very distorted waves an analysis is absolutely essential), and all the above quantities may be found more accurately and more expeditiously from the analysis than from the laborious method of working from a large number of measured ordinates. Numerical examples of the use of formulas (14), (15), and (16) will be given below in Examples 9, 10, and 11.

3. DERIVATION OF THE EQUATIONS OF POWER CURVES

The equation of the curve showing the *instantaneous power* during the cycle follows without difficulty from the equations of the emf. and current curves.

Writing

$$e = E_1 \sin (pt - \theta_1) + E_3 \sin (3pt + \theta_3) + \ldots + E_n \sin (npt - \theta_n) + \ldots$$

$$i = I_1 \sin (pt - \delta_1) + I_3 \sin (3pt - \delta_3) + \ldots + I_n \sin (npt - \delta_n) + \ldots$$

we have for the power

$$\begin{aligned} ei = {} & E_1I_1 \sin(pt-\theta_1)\sin(pt-\delta_1) + E_1I_n \sin(pt-\theta_1)\sin(npt-\delta_n) + \ldots \\ & E_3I_1 \sin(3pt-\theta_3)\sin(pt-\delta_1) + \ldots \\ & + E_3I_n \sin(3pt-\theta_3)\sin(npt-\delta_n) + \ldots \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & E_nI_1 \sin(npt-\theta_n)(pt-\delta_1) + \ldots \\ & + E_nI_n \sin(npt-\theta_n)\sin(npt-\delta_n) + \ldots \end{aligned} \qquad (17)$$

Using the relation $2 \sin x \sin y = \cos(x-y) - \cos(x+y)$ the equation (17) falls into the following

$$\begin{aligned} 2ei = {} & E_1I_1 \cos(\delta_1-\theta_1) + E_3I_3 \cos(\delta_3-\theta_3) + \ldots \\ & + E_nI_n \cos(\delta_n-\theta_n) + \ldots \\ & + \text{terms involving the cosines of even multiples of } pt. \end{aligned}$$

The first line of the preceding equation gives the average power, as has already been shown. The cosine terms are of the form $P_n \cos[2npt-\alpha_1]$, $Q_n \cos[2npt-\alpha_2]$ etc. These terms can be simplified by using the relation

$$\begin{aligned} & M_1 \cos(x-\alpha_1) + M_2 \cos(x-\alpha_2) + M_3 \cos(x-\alpha_3) + \ldots \\ & \qquad = N \sin(x-\gamma) \end{aligned} \qquad (17a)$$

where

$$\begin{aligned} N^2 = {} & (M_1 \sin\alpha_1 + M_2 \sin\alpha_2 + M_3 \sin\alpha_3 + \ldots)^2 + (M_1 \cos\alpha_1 \\ & + M_2 \cos\alpha_2 + M_3 \cos\alpha_3 + \ldots)^2 \end{aligned}$$

and

$$\tan\gamma = \frac{-(M_1 \cos\alpha_1 + M_2 \cos\alpha_2 + M_3 \cos\alpha_3 + \ldots)}{(M_1 \sin\alpha_1 + M_2 \sin\alpha_2 + M_3 \sin\alpha_3 + \ldots)}$$

The equation for the power then reads

$$\begin{aligned} 2ei = 2P = {} & E_1I_1 \cos(\delta_1-\theta_1) + E_3I_3 \cos(\delta_3-\theta_3) + \ldots \\ & + E_nI_n \cos(\delta_n-\theta_n) + \ldots \\ & + N_1 \sin(2pt-\gamma_1) + N_2 \sin(4pt-\gamma_2) \\ & + N_3 \sin(6pt-\gamma_3) + \ldots \end{aligned} \qquad (18)$$

This procedure will be illustrated by Example 12, below. In those cases where the higher harmonics in both the current and emf. wave are not negligible, the number of terms which must be included in equation (17) becomes large, and the method becomes

more laborious. In such a case it may often be easier to form the products of the ordinates used in the analysis of the emf and current curves, and carry through the analysis of the power curve using the schedules for the analysis of curves in which even harmonics are included. (See Appendix B.) Both methods are illustrated in Example 12, and are seen to give essentially the same equation, as should be the case.

4. RESOLUTION OF THE POWER CURVE INTO TWO COMPONENTS.

It is often of interest to resolve the power curve into two components, one representing the power lost in the circuit in such ways as in heating a resistance, in iron losses, or in dielectric losses, and a second showing the rate at which energy is alternately stored in and recovered from the magnetic field or the dielectric. The average value of the power *dissipated* is $EI \cos \phi$, where E and I are, respectively, the effective values of the emf and the current, and $\cos \phi$ is the power factor, the angle ϕ being in the case of sine waves the angle of phase difference between the emf. and the current.

In the case of sine waves it is mathematically the same, as far as the value of the average power is concerned, whether we resolve the emf. into two components $E \cos \phi$ and $E \sin \phi$, respectively in phase and in quadrature with the current, and take the products of these components with the current, or whether we consider the components of the power to be the result of taking the products of the emf. with the two currents $I \cos \phi$ and $I \sin \phi$, found by resolving the current along the emf. vector and at right angles to it, respectively.

Thus, it is customary to speak of the *power* and *wattless* components of the current as well as of the corresponding quantities for the emf. Logically, however, it does not seem to be correct to speak of a wattless component of the current, since in the power loss I^2r, not only the "power" component $I \cos \phi$, but also the component $I \sin \phi$ of the current enters. Further, it is easy to show that of these two methods of resolution only that where the emf. is split up into power and wattless components gives a correct picture of the facts of the case. However, since it is very often conven-

ient to make use of the conceptions of the "power" and "wattless" components of the current, as defined above (as, for example, in the usual treatment of the theory of the magnetizing current of a transformer), both methods will be treated below, after we have first considered wherein the two methods of treatment differ.

Assuming the emf and current to vary according to a simple sine law, we may put for the instantaneous values of emf and current

$$\begin{aligned} e &= E \sin pt \\ i &= I \sin (pt - \phi) \end{aligned} \tag{18a}$$

and the instantaneous value of the power is given by

$$ei = \frac{1}{2} EI \left[\cos \phi - \cos (2pt - \phi)\right] \tag{19}$$

The average value of the second term is zero; the first term gives the usual expression for the average value of the power, i. e., the average value of the power *dissipated.*

Let us consider the case of an inductive circuit; entirely similar relations follow if capacity is also assumed to be present. Resolving the emf. into two components, e_1 and e_2, respectively in phase with the current, and 90° ahead of it in phase, we may write

$$\begin{aligned} e_1 &= E \cos \phi \sin (pt - \phi) \\ e_2 &= E \sin \phi \cos (pt - \phi) \end{aligned} \tag{20}$$

and thence,

$$\begin{aligned} e_1 i &= \frac{1}{2} EI \cos \phi \left[1 - \cos (2pt - 2\phi)\right] \\ e_2 i &= \frac{1}{2} EI \sin \phi \sin (2pt - 2\phi) \end{aligned} \tag{21}$$

The sum of these two component power curves gives the same expression for the total power as in equation (19).

If we resolve the current into two components, i_1 and i_2, respectively in phase with the emf and 90° behind it in phase, we find

$$\begin{aligned} i_1 &= I \cos \phi \sin pt \\ i_2 &= -I \sin \phi \cos pt \end{aligned} \tag{22}$$

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and, accordingly

$$\begin{aligned} ei_1 &= \frac{1}{2}EI\cos\phi[1-\cos 2pt] \\ ei_2 &= \frac{1}{2}EI\sin\phi\sin 2pt \end{aligned} \tag{23}$$

The sum of these two components gives the same equation (19) for the total power as before.

The power equation for an inductive circuit may be written

$$ei = Ri^2 + \frac{d}{dt}\left(\frac{1}{2}Li^2\right) \tag{24}$$

the first term giving the rate at which energy is being dissipated, the second term the rate at which the kinetic energy of the current is being changed.

Substituting the value of i from (18a), we find

$$\begin{aligned} Ri^2 &= \frac{1}{2}I^2R\,[1-\cos(2pt-2\phi)] \\ &= \frac{1}{2}EI\cos\phi[1-\cos(2pt-2\phi)] \end{aligned} \tag{25}$$

since the equation $I = \frac{E}{R}\cos\phi$ exists between the maximum values of current and emf.

The second term gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(Li^2) &= \frac{1}{2}LpI^2\sin(2pt-2\phi) \\ &= \frac{1}{2}EI\sin\phi\sin(2pt-2\phi) \end{aligned} \tag{26}$$

since $LpI = E\sin\phi$.

The two component curves of instantaneous power in equation (21), found by resolving the emf. into an in-phase component and a quadrature component, with respect to the current, are thus seen to represent correctly at every instant, the values of the power expended in Joule heat and the power stored in the magnetic field, equations (25) and (26).

If we now substitute the relation $i = i_1 + i_2$ in (24), the power equation may be written

$$ei = R(i_1^2 + 2i_1 i_2 + i_2^2) + L\left(i_1\frac{di_1}{dt} + i_2\frac{di_2}{dt} + i_1\frac{di_2}{dt} + i_2\frac{di_1}{dt}\right) \tag{27}$$

and, using the values of i_1 and i_2 from (22), it is easy to show that the two components of the power ei_1 and ei_2 in equation (23) are given by the two sums of terms $R(i_1^2 + i_2^2) + L\left(i_1\frac{di_2}{dt} + i_2\frac{di_1}{dt}\right)$ and $2Ri_1 i_2 + L\left(i_1\frac{di_1}{dt} + i_2\frac{di_2}{dt}\right)$ respectively, which together make up (27).

Remembering that the phases of $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$ are 90° ahead of i_1 and i_2, respectively, it appears that the first series of terms consists of products of pairs of in-phase quantities, while the second series consists of products of quantities in quadrature with one another. The separation of those terms in the equation of the power which, on the average, contribute to the average power, from those terms whose average value taken over the cycle is zero, is therefore correctly carried out, but this resolution of the current must nevertheless be regarded as artificial rather than as representing the actual physical relations.

Passing to the consideration of nonharmonic waves, let us write, as before,

$$\begin{aligned} e &= E_1 \sin(pt - \theta_1) + E_3 \sin(3pt - \theta_3) + E_5 \sin(5pt - \theta_5) + \dots \\ i &= I_1 \sin(pt - \delta_1) + I_3 \sin(3pt - \delta_3) + I_5 \sin(5pt - \delta_5) + \dots \end{aligned} \tag{28}$$

Remembering that the product of any harmonic of the emf wave by any harmonic of different frequency in the current wave contributes nothing to the average value of the power taken over the cycle, it is logical to write for the power and wattless components of the emf the following equations of definition

$$\begin{aligned} e_1 &= E_1 \cos(\delta_1 - \theta_1) \sin(pt - \delta_1) + E_3 \cos(\delta_3 - \theta_3) \sin(3pt - \delta_3) + \dots \\ e_2 &= E_1 \sin(\delta_1 - \theta_1) \cos(pt - \delta_1) + E_3 \sin(\delta_3 - \theta_3) \cos(3pt - \delta_3) + \dots \end{aligned} \tag{29}$$

For the components of the current in phase and in quadrature with the emf, we have the similar equations

$$i_1 = I_1 \cos(\delta_1 - \theta_1) \sin(pt - \theta_1) + I_3 \cos(\delta_3 - \theta_3) \sin(3pt - \theta_3) + \dots$$

$$-i_2 = I_1 \sin(\delta_1 - \theta_1) \cos(pt - \theta_1) + I_3 \sin(\delta_3 - \theta_3) \cos(3pt - \theta_3) + \dots \quad (30)$$

If the harmonics E_1, E_3, . . . and I_1, I_3, . . . of the emf. and current are already known from previous analysis of the curves, it is evidently a very simple matter to write down the equations of these component emfs and currents, and these may be used to derive the equations of the component power curves as will be shown below. If, however, the form of the curves given in these equations is of interest, it is necessary to plot these equations, which is a laborious undertaking. In case the harmonics are mostly all small, or negligible, the matter is simplified, since the harmonic waves may be drawn accurately enough without calculation. For this the zero points and the positions of the maxima and minima are plotted and the rest of the curves drawn in free hand. The resultant curve of these harmonic waves may then be determined graphically, and its ordinates show the deviation of the actual curve from the simple sine wave of the fundamental. The latter may be plotted carefully, and corrected by means of the curve of deviations, to obtain a sufficiently accurate representation of the actual curve.

5. DERIVATION OF THE EQUATIONS OF COMPONENT POWER CURVES

The equations for e_1i and e_2i are found by multiplying the expressions (29) by the equation (28) for i. In the case of the component e_1i there will also be, in addition to the series of constant terms which express the average power (see equation (16)), a series of terms of the form $P \sin(mpt - \delta_m) \cdot \sin(npt - \delta_n)$, which may be resolved by the relation $2 \sin(mpt - \delta_m) \sin(npt - \delta_n) = \cos[(m-n)pt - (\delta_m - \delta_n)] - \cos[(m+n)pt - (\delta_m + \delta_n)]$, and finally all the terms involving each frequency are to be collected into a single term involving the sine of the same frequency, by means of equations given already.

For the component e_2i we have terms of the form $Q \sin (mpt - \delta_m) \cos (npt - \delta_n)$, which may be resolved by the relation $2 \sin (mpt - \delta_m) \cos (npt - \delta_n) = \sin [(m+n)pt - (\delta_m + \delta_n)] - \sin [(m-n)pt - (\delta_m - \delta_n)]$, and then all the terms of the same frequency may be collected by means of the following transformation equations:

$$M_1 \sin (2npt - \alpha_1) + M_2 \sin (2npt - \alpha_2) + M_3 \sin (2npt - \alpha_3) + \ . \ . \ . = N \sin (2npt - \gamma) \tag{31}$$

where

$$N^2 = [M_1 \sin \alpha_1 + M_2 \sin \alpha_2 + M_3 \sin \alpha_3 + \ . \ . \ . \ ]^2 + [M_1 \cos \alpha_1 + M_2 \cos \alpha_2 + M_3 \cos \alpha_3 + \ . \ . \ . \ ]^2 \tag{32}$$

$$\tan \gamma = \frac{M_1 \sin \alpha_1 + M_2 \sin \alpha_2 + M_3 \sin \alpha_3 + \ . \ . \ .}{M_1 \cos \alpha_1 + M_2 \cos \alpha_2 + M_3 \cos \alpha_3 + \ . \ . \ .}$$

The problem of finding the equations of the curves for ei_1 and ei_2 presents nothing essentially different from the foregoing considerations.

The details of these calculations will be clearer from a consideration of Examples 13*a* and 13*b*, in which are illustrated all the steps involved in the calculation of the equations of the curves of the power components.

The amount of labor necessary for the calculations rapidly increases with the number of harmonics present in the waves of emf and current, and if it were not for the amount of work necessary to calculate ordinates from the curves of the emf. or current components, it would be easier to actually carry through the analysis of the curves from a set of calculated ordinates as is done in Example 12 to obtain the equation of the total power curve.

VI. EXAMPLES ILLUSTRATING THE PRECEDING SECTION

EXAMPLE 9. CALCULATION OF THE AVERAGE VALUES OF CURRENT AND EMF. WAVES, FORMULA 14

To illustrate the formulas, we will consider the waves of emf. and current shown in Fig. 3, curves A and B. The details of the calculation of the average values are shown in Table 12. In the first column is given the order of the harmonic; in the second, the corresponding amplitudes found by analysis; in the third, the

values of the phase angles ψ_n, as derived by analysis (Example 3). From these are calculated the values of $(\psi_n - \psi_1)$, and in column

TABLE 12

Calculation of the Average Values of Current and Electromotive Force Waves

ELECTROMOTIVE FORCE

	E_n	ψ_n	$\psi_n-\psi_1$	β_n	Cos β_n	$\frac{E_n}{n}$	Products
1	33.217	1°.6	0	0	1.0000	33.217	33.217
3	0.970	2.1	0°5	1°5	0.9997	0.323	0.323
5	1.702	41.7	40.1	200.5	−0.9367	0.340	− 0.319
7	0.218	38.3	36.7	256.9	−0.2267	0.031	− 0.007
9	0.355	2.2	0.6	5.4	0.9956	0.039	0.039
11	0.205	19.3	17.7	194.7	−0.9673	0.019	− 0.018
13	1.251	3.8	2.2	28.6	0.8780	0.096	0.084
15	1.528	15.7	14.1	211.5	−0.8526	0.102	− 0.087
17	0.159	3.0	1.4	23.8	0.9150	0.009	0.008
			$33.240\times\frac{2}{\pi}=21.16$				33.671
							− .431
			Average value of emf.			Sum	33.240

CURRENT

	I_n	ψ_n	$\psi_n-\psi_1$	β_n	Cos β_n	$\frac{I_n}{n}$	Products
1	29.627	66°.9	0	0	1.0000	29.627	29.627
3	3.866	27.8	−39.1	−117.3	−0.4586	1.289	− 0.592
5	0.459	26.2	−40.7	−203.5	−0.9171	0.092	− 0.084
7	0.211	15.3	−51.6	−361.2	0.9998	0.030	0.030
9	0.155	11.6	−55.3	−497.7	−0.7396	0.017	− 0.013
11	0.163	15.4	−51.5	−566.5	−0.8949	0.015	− 0.014
13	0.214	9.9	−57.0	−741.0	0.9336	0.016	0.015
15	0.188	15.3	−51.6	−774.0	0.5878	0.013	0.008
17	0.052	18.1	−48.8	−829.6	−0.3355	0.003	− 0.001
			$28.976\times\frac{2}{\pi}=18.45$				29.680
							− .704
			Average value of current.			Sum	28.976

5 the values of $\beta_n = n(\psi_n - \psi_1)$. Columns 6 and 7 give cos β_n and $\frac{E_n}{n}$, and in the last column appear the products $\frac{E_n}{n}$ cos β_n.

The sum of these latter, multiplied by $\frac{2}{\pi}$, gives the average ordinate of the emf wave as 21.16.

The second half of the table relates to the current wave, and the preceding explanation will make this clear. The average ordinate of the current was found to be 18.45.

As a check on these calculations the average values were also determined by actually measuring and summing the ordinates of each dot (phase difference of 2°) for half a cycle, taking as the origin the point where the curve crosses the axis. The values found were 21.29 for the emf and 18.60 for the current. The differences between these values and those found from the results of analysis of the curves are probably to be explained by the fact that only one-half of the wave was measured, while the analyses rest upon the average ordinates of a whole wave.

Similar calculations were made for the condenser current and emf. (curves B and A, Fig. 5), which have already been analyzed in Example 7. The condenser current is so irregular, crossing the axis so often, that one has to depend on the analysis to tell where the origin of phase should be taken to agree with our definition on page 608. The results of the two methods are given below.

	Calculated	Measured ordinates
Average emf	21.86	21.84
Average current	13.88	13.56

The labor of measuring all these ordinates and taking their sum is, of course, vastly greater than that of calculating the average value from the results of the curve analysis.

EXAMPLE 10. CALCULATION OF THE EFFECTIVE VALUES OF EMF. AND CURRENT, FORMULA 15

The same curves were treated as in the preceding example. Details of the calculation are given in Table 13.

TABLE 13

Calculation of the Effective Values of Current and Electromotive Force Waves

	Transformer Circuit				Condenser Circuit			
n	E_n	E_n^2	I_n	I_n^2	E_n	E_n^2	I_n	I_n^2
1	33.217	1,103.36	29.627	877.76	35.01	1,225.70	22.84	521.67
3	0.970	.94	3.866	14.95	0.99	0.98	2.28	5.20
5	1.702	2.90	0.459	0.21	1.90	3.61	7.22	52.13
7	0.218	.05	0.211	0.04	1.93	3.72	11.48	131.79
9	0.355	.13	0.155	0.02	0.09	0.01	1.51	2.28
11	0.205	.04	0.163	0.03	0.45	0.20	7.17	51.41
13	1.251	1.56	0.214	0.05	0.18	0.03	0.28	0.08
15	1.528	2.34	0.188	0.04	0.08	0.01	0.85	0.72
17	0.159	.03	0.052	0.00	0.03	0.00	0.66	0.44
	Sum	1,111.35	Sum	893.10	Sum	1,234.26	Sum	765.72
	Sum÷2=	555.68	Sum÷2=	446.55	Sum÷2=	617.13	Sum÷2=	382.86
	Effective value	*23.57*		*21.13*		*24.84*		*19.56*
	Value found from measured ordinates	*23.70*		*21.25*		*24.85*		*18.97*

In the first column is given the order of the harmonic, in the second the amplitudes of the various harmonics, as found by the analysis (Examples 3 and 7), and in the third column the squares of the amplitudes. Half the sum of the squares gives the square of the effective value.

As a check on these calculations, the effective values were also calculated by averaging the squares of the ordinates, already measured for calculating the average values in the previous example. The results thus found are also given in the table, and it will be seen that the agreement is satisfactory when it is considered that ordinates were measured for only one-half wave. In the case of the condenser current the changes of the current are so abrupt that the measured ordinates are apparently not close enough together to give a very accurate value.

For completeness the form factors of these curves are appended, as found both by calculation from the results of the analyses and as obtained from the measured ordinates.

Method	Transformer		Condenser	
	Emf	Current	Emf	Current
Analysis	1.114	1.145	1.137	1.409
Measured ordinates	1.113	1.1425	1.137	1.399

EXAMPLE 11. CALCULATION OF THE AVERAGE POWER, POWER FACTOR, AND EFFECTIVE PHASE DIFFERENCE, FORMULA 16

The same curves are considered here as in the preceding examples. In the first column of Table 14 is given the order of the harmonics. The second and third columns contain the amplitudes of the harmonics of the emf, and current, respectively, curves *A* and *B*, Fig. 3, while in the fourth and fifth appear the corresponding phase angles θ_n and δ_n, found by analysis. In column 6 are the calculated values of $(\delta_n - \theta_n)$, and in column 7

TABLE 14

Calculation of the Average Power from the Results of the Analyses of Current and Emf. Waves

n	E_n	I_n	θ_n	δ_n	$\delta_n-\theta_n$	$\cos(\delta_n-\theta_n)$	E_nI_n	Products
			°	°	°			
1	33.217	29.627	1.6	66.9	65.3	0.4179	984.12	411.24
3	0.970	3.866	6.3	83.5	77.2	0.2215	3.75	0.83
5	1.702	0.459	208.8	131.0	− 77.8	0.2113	0.78	0.17
7	0.218	0.211	268.1	107.1	−161.0	−0.9455	0.05	− 0.04
9	0.355	0.155	20.1	104.6	84.5	0.0958	0.06	0.00
11	0.205	0.163	211.8	169.7	− 42.1	0.7420	0.03	0.02
13	1.251	0.214	49.3	129.1	79.8	0.1771	0.27	0.05
15	1.528	0.188	235.6	229.1	− 6.5	0.9936	0.29	0.29
17	0.159	0.052	51.1	308.0	256.9	−0.2250	0.01	− 0.00
							Sum	412.56
							Average power	*206.28*

Effective emf............ 23.57
Effective current........ 21.13
Power factor = 206.28 ÷ (23.57 × 21.13) = *0.4142* = cos *65°5*

the cosines of these angles. The eighth and ninth columns give the products of the amplitudes and the power $E_n I_n \cos(\delta_n - \theta_n)$. The sum of these latter values, divided by 2, is the average power. The power factor is given by the quotient of this calculated value

of the average power by the product of the effective values of current and emf. found in the preceding example. The value found is 0.4142, which corresponds to an effective phase difference of 65°.5, a value not very different from the phase difference of 65°.3 existing between the fundamentals of current and emf.

The average power calculated directly from the measured ordinates of the emf and current waves is 212.7, which gives a power factor of 0.4223, corresponding to an effective phase difference of 65°.0.

Similar calculations were also made for the case of the condenser circuit considered in the preceding examples. As calculated from the equations of the curves of emf and current (see Example 7) the average power is 89.37, indicating a power factor of 0.1839 and an effective phase difference of 79°.4.

The scale of the emf curve, Fig. 5, was 20 divisions = 82 volts, while for the current 20 divisions = 0.71 ampere. The calculated values of the effective emf and current, expressed in divisions in Table 13, were, respectively, 101.7 volts and 0.637 ampere. The average power is therefore $89.37 \times 0.0355 \times 4.1$ or 13.03 watts, which corresponds to an effective resistance of $\frac{13.03}{(0.637)^2}$ or 27.0 ohms. This includes the equivalent resistance which represents the absorption of the condenser, as well as the actual resistance in series with the condenser. The capacity was about 9 mf.

By direct calculation from the ordinates of the emf and current curves the average power comes out as 79.57, which corresponds to a power factor of $0.1688 = \cos 80°.3$. Reducing to watts we have 11.58 watts, which indicates an effective resistance of 25.6 ohms.

The difference between the results by the two methods of calculation was explained when, on closer examination of the curves, it was noticed that the corresponding points of the current and emf. curves, used in the analyses, do not have exactly the same phase, but were given a displacement in the drawing of the curve of about 0.4 the phase difference of successive points, or 0°.8 in phase, and in such a direction as to make the phase difference found by analysis (on the assumption that corresponding ordi-

nates have exactly the same phase) come out too small. Applying this correction the difference between the results by the two methods almost disappears. Attention is thus called to a source of error which must be guarded against if the results of the analyses are to be applied to the calculation of power factor.

EXAMPLE 12. DERIVATION OF THE EQUATION OF A POWER CURVE

In this example will be treated the derivation of the curve of instantaneous power for the emf and current curves, Fig. 3, already investigated in the previous examples. These are the curves of the magnetizing current in a transformer, at such a frequency that the magnetic induction was about double the normal value, together with the impressed emf causing this current to flow.

The problem will here be treated by two methods, (*a*) calculation from the results of the analysis of the emf and current waves, and (*b*) direct analysis, using as fundamental ordinates the products of the ordinates used in the analysis of the waves of emf and current.

(*a*) **First Method.**—The author's experience has led to the arrangement of the calculation exemplified in Table 15.

Columns 2 to 5 give the amplitudes and phase angles found in the analyses (Example 3) and the remainder of the first part of the table contains the separate products of the harmonic amplitudes of the current and emf, taken two at a time; thus the product of the ninth harmonic of the emf by the eleventh harmonic of the current appears in the column headed E_9 and in the row in which appears 11. It will be noticed that a good many of these products are, relatively, negligibly small, except in such cases as very high precision is desired. The labor of calculation may, therefore, in a good many cases be materially lessened by the omission of those terms which are of small importance.

In the second part of the table is illustrated the calculation of the amplitude and phase of one of the harmonics of the power curve—that harmonic which has a frequency of sixteen times the frequency of the fundamental of the current and emf.

In the third column are given those products $E_m I_n$, given in the first part of the table, for which the sum of the orders m and n of the harmonics is equal to the order of the desired harmonic; in this case 16. In the fourth column is given the sum of the phase angles, δ_n and θ_m of the current and emf harmonics, for

TABLE 15

Details of the Calculation of the Equation of a Power Curve from the Results of the Analysis of the Emf. and Current Curves

n	E_m	I_n	θ_m	δ_n	Products involving—								
					E_1	E_3	E_5	E_7	E_9	E_{11}	E_{13}	E_{15}	E_{17}
1	33.217	29.627	1°6	66°9	984.12	28.74	50.43	6.46	10.51	6.07	37.06	45.27	4.71
3	0.970	3.866	6.3	83.5	128.42	3.75	6.58	0.84	1.37	0.79	4.83	5.91	0.61
5	1.702	0.459	208.8	131.0	15.25	0.45	0.78	0.10	0.16	0.09	0.58	0.70	0.07
7	0.218	0.211	268.1	107.1	7.01	0.20	0.36	0.05	0.08	0.04	0.26	0.32	0.03
9	0.355	0.155	20.1	104.6	5.15	0.15	0.26	0.03	0.06	0.03	0.19	0.24	0.02
11	0.205	0.163	211.8	169.7	5.41	0.16	0.28	0.04	0.07	0.03	0.20	0.25	0.03
13	1.251	0.214	49.3	129.1	7.11	0.21	0.36	0.05	0.08	0.04	0.27	0.33	0.03
15	1.528	0.188	235.6	229.1	6.24	0.18	0.32	0.04	0.07	0.04	0.24	0.29	0.03
17	0.159	0.052	51.1	308.0	1.73	0.05	0.09	0.01	0.02	0.01	0.06	0.08	0.01

CALCULATION OF TERM IN 16 *pt.*

m	n	$M=E_m I_n$	Phase Angle a		sin a	cos a	M sin a	M cos a
			$\delta_n+\theta_m$	$\pm(\delta_n-\theta_m)$				
1	15	− 6.24	230°7		−0.7738	−0.6334	4.83	3.96
1	17	1.73		306°4	−0.8049	0.5934	− 1.40	1.03
3	13	− 0.21	135.4		0.7022	−0.7120	− 0.15	− 0.15
5	11	− 0.28	378.5		0.3173	0.9483	− 0.09	− 0.27
7	9	− 0.03	372.7		0.2198	0.9755	− 0.01	− 0.03
9	7	− 0.08	127.2		0.7965	−0.6046	− 0.06	0.05
11	5	− 0.09	342.8		−0.2940	0.9553	0.03	− 0.09
13	3	− 4.83	132.8		0.7337	−0.6794	− 3.54	3.28
15	1	−45.27	302.5		−0.8434	0.5373	38.18	−24.32
17	1	4.71		− 15.8	−0.2723	0.9622	− 1.28	4.52
						Sums	36.51	−12.02

$$N=\sqrt{36.51^2+12.02^2}=38.38$$

$$\tan\gamma=\frac{12.02}{36.51} \qquad \gamma=18°2$$

those cases in which the sum of the orders of the harmonics of emf and current is to be taken. Similarly, the fifth column contains the values of the differences of the phase angles of the

harmonics of the current and emf corresponding to those cases where the differences of the indices m and n is equal to 16.

From the formula

$$2 \sin (mpt-\theta_m) \sin (npt-\delta_n) = \cos [(n-m)pt-(\delta_n-\theta_m)] \\ -\cos [(n+m)pt-(\delta_n+\theta_m)]$$

we see that the products $E_m I_n$, taken from the first part of the table, are to be given the positive or negative sign, according as the difference of n and m has to be taken to get 16, or the sum. Thus, in the first term $(n+m)=15+1$, and the negative sign is given to the product $E_1 I_{15}=6.24$; in the second term $(n-m)=17-1$ and the product $E_1 I_{17}=1.73$ is to be given the positive sign. With regard to the phases, we take the sum $(\delta_n+\theta_m)$ when $(m+n)=16$, and if $(n-m)=16$ the phase is $(\delta_n-\theta_m)$; if $(m-n)=16$, then $(\theta_m-\delta_n)$ is to be taken, hence the heading of the fifth column is given the double sign, with the understanding that the sign is to be taken as positive or negative according as n is greater or less than m.

In the sixth and seventh columns are given the sines and cosines of the phase angle, as just defined, and in the eighth and ninth columns the product of these functions with the corresponding values of $E_m I_n$. The derivation of the resultant amplitude of the harmonic from these quantities $M \sin \alpha$ and $M \cos \alpha$ is clearly indicated in the table, and is in accordance with equation (17*a*). The term in $16pt$ as thus found is $38.4 \sin (16pt-18°2)$.

Proceeding in this manner for each harmonic, the complete equation, including terms in $18pt$, is

$$\begin{aligned} 2P=2ei=412.56 &+873.9 \sin (2pt-157°.6)+132.3 \sin (4pt-148°.3) \\ &+38.5 \sin (6pt-2°.6)+0.9 \sin (8pt-155°.2) \\ &+13.7 \sin (10pt-155°.0) \\ &+30.1 \sin (12pt-271°.9)+43.7 \sin (14pt-141°.2) \\ &+38.4 \sin (16pt-18°.2)+3.26 \sin (18pt-67°.1) \end{aligned}$$

The data in the Table 15 suffices to allow the equation to be extended to include terms in $34pt$. It is evident, however, from an inspection of the minuteness of the products $E_m I_n$, which would be used in calculating these higher terms, that all terms above that in $18pt$ must be practically negligible.

(*b*) **Second Method—Determination of the Equation of the Power Curve by Direct Analysis.**—For this purpose, the products were taken of the fundamental ordinates of the emf. curve, used in the analysis, by the corresponding ordinates which formed the basis of the analysis of the current curve. For convenience in plotting, these products were divided by 20 and were then used as fundamental ordinates in an analysis according to Schedule 3 in Appendix B. This schedule is for the analysis of a curve in which even harmonics and a constant term are included. Since 18 fundamental ordinates are required, the analysis gives, in addition to the constant term, the harmonics up to and including the *cosine* term of nine times the frequency of the fundamental. The term in the *sine* of nine times the frequency of the fundamental requires that a second set of ordinates lying halfway between those of the first set be measured, but the amplitude and phase of this term follow then immediately without further calculation.

The products of the emf. and current ordinates, used in this analysis, are spaced 10° apart in terms of the phase of the fundamental frequency of the emf and current, but since the fundamental of the power curve has twice the frequency of the fundamental of emf. and current, these 18 ordinates extend over the complete cycle of the power curve, and, therefore, fulfill the required conditions of the above scheme of analysis.

The detailed analysis of this power curve appears in Table 16 and illustrates the use of the Schedule 3, Appendix B. The work differs only in minor particulars from that in the case of the analysis of emf. and current curves in the Examples 1, 2, and 3.

The equation found by analysis is compared in Table 17 with that found by the first method of calculation, the amplitudes in this case having been divided by 20 to reduce to the same scale as that used in the analysis. The first column gives the frequency of the harmonic in terms of that of the fundamental of the emf. and current waves. The second and fourth columns give the amplitudes and phases found by the first method, the third and fifth the same quantities as given by the analysis.

The two curves are practically the same, as they should be, since they depend on the same data, namely, the results of the analysis of the emf. and current curves, or, what is the same

	Ordinates		s	d	
0	− 0.15		− 0.15	− 0.15	
1	− 6.65	10.1	+ 3.45	−16.75	sin
2	−11.55	17.55	6.0	−29.1	sin
3	−12.0	27.05	15.05	−39.05	sin
4	−12.0	32.9	20.9	−44.9	sin
5	− 6.3	31.3	25.0	−37.6	
6	+ 0.2	33.0	33.2	−32.8	
7	8.0	23.7	31.7	−15.7	
8	10.7	21.55	32.25	−10.85	
9	18.3		18.3	+18.3	

$\Sigma_0 = -0.15 + 33.2 = 33.05$

$\Sigma_1 = 28.45 + 31.7 = 60.15$ sin

$\Sigma_2 = 26.9 + 32.25 = 59.15$ sin

$\Sigma_3 = 15.05 + 18.3 = 33.35$ sin

sin

∂_0 92.20 sin

∂_1 93.50

$\triangle_1 = -32.45 + 37.6 = 5.15$

$\triangle_2 = -39.95 + 44.9 = 4.95$

s_0	+	−
$B_0 + B_9 =$	10.244	
$B_1 + B_8 =$		8.629
$B_2 + B_7 =$		2.365
$B_3 + B_6 =$	0.772	
$B_4 + B_5 =$		0.173
	11.016	11.167
$s_0 =$		− 0.151
Actual		− 0.150

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TABLE 16

Analysis of Power Curve from Eighteen Ordinates

	Ordinates		s	d		1 and 8				2 and 7				3 and 6				4 and 5			
0	− 0.15		− 0.15	− 0.15		÷	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
1	− 6.65	10.1	+ 3.45	−16.75	sin 20°		5.729		3.711	12.860			15.357					5.369			9.953
2	−11.55	17.55	6.0	−29.1	sin 40°		10.092		18.705		10.767	6.974							24.169	28.860	
3	−12.0	27.05	15.05	−39.05	sin 60°		33.818		28.405		33.818	28.405		4.460		4.267		33.818			28.405
4	−12.0	32.9	20.9	−44.9	sin 80°		37.029		44.217	15.461			28.657						16.496	10.685	
5	− 6.3	31.3	25.0	−37.6			− 86.668		−95.038	28.321	44.585	35.379	44.014	4.460		4.287		39.187	40.665	39.545	38.358
6	+ 0.2	33.0	33.2	−32.8			− 95.038				−16.264		− 8.635	4.287					− 1.478	1.187	
7	8.0	23.7	31.7	−15.7			−181.706		8.370		−24.899		− 7.629	8.747		0.173			− 0.291		− 2.665
8	10.7	21.55	32.25	−10.85		$A_1=$	−20.190	$A_8=$	0.930	$A_2=$	− 2.767	$A_7=$	− 0.848	$A_3=$	0.972	$A_6=$	0.019	$A_4=$	− 0.032	$A_5=$	− 0.296
9	18.3		18.3	+18.3																	

		1 and 8				2 and 7				3 and 6				4 and 5				0 and 9	
$\Sigma_0=-0.15+33.2=33.05$		÷	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−		
$\Sigma_1=28.45+31.7=60.15$	sin 10°	3.629			4.341	1.042		5.505						5.600		0.599			
$\Sigma_2=26.9+32.25=59.15$	sin 30°		16.600	7.525			16.600		7.525		29.575	30.075			16.600		7.525		
$\Sigma_3=15.05+18.3=33.35$	sin 50°	4.596			24.293	24.704		2.643						16.010		19.151			
	sin 70°		30.305	3.242			19.640		23.492						5.638		29.789		
δ_0 92.20	sin 90°		0.150		18.300		0.150	18.300		33.050			33.350		0.150	18.300		92.200	93.500
δ_1 93.50		8.225	47.055	10.767	46.934	25.746	36.390	26.448	31.017	33.050	29.575	30.075	33.350	21.610	22.388	38.050	37.314	92.200	93.500
			−38.830		−36.167		−10.644		− 4.569	3.475			− 3.275		− 0.778	0.736		185.700	−1.300
$\triangle_1=-32.45+37.6=5.15$			−74.997		− 2.663		−15.213		− 6.075	0.200		6.750			− 0.042		− 1.514	92.850	−0.650
$\triangle_2=-39.95+44.9=4.95$		$B_1=$	− 8.333	$B_8=$	− 0.296	$B_2=$	− 1.690	$B_7=$	− 0.675	$B_3=$	0.022	$B_6=$	0.750	$B_4=$	− 0.005	$B_5=$	− 0.168	$B_0=$ 10.316	B_9 −0.072

CHECKS

s_0	+	−
$B_0+B_9=$	10.244	
$B_1+B_8=$		8.629
$B_2+B_7=$		2.365
$B_3+B_6=$	0.772	
$B_4+B_5=$		0.173
	11.016	11.167
$s_0=$		− 0.151
Actual		− 0.150

s_9	+	−
$B_0-B_9=$	10.388	
$-(B_1-B_8)=$	8.037	
$B_2-B_7=$		1.015
$-(B_3-B_6)=$	0.728	
$B_4-B_5=$	0.163	
	19.316	1.015
$s_9=$	18.301	
Actual =	18.300	

d_3	+	−
(A_1+A_8)		19.260
(A_2+A_7)		3.615
$-(A_4+A_5)$	0.328	
Sums	0.328	22.875
		−22.547
×2		−45.094
× sin 60°		−38.971
		.081
d_3		−39.052
Actual		−39.05

d_6	+	−
(A_1-A_8)		21.120
$-(A_4-A_7)$	1.918	
(A_4-A_5)	0.264	
Sums	2.162	21.120
		−18.938
×2		−37.876
× sin 60°		−32.043
		.753
		.005
d_6		−32.801
Actual		−32.8

(A_3+A_6) and (A_3-A_6)	+	−
$(d_1+d_2)=$		45.85
$-(d_4+d_5)$	82.5	
(d_7+d_8)		26.55
Sums	82.5	72.40
	10.10	
× sin 60°	8.747	
÷9	0.972 = A_3	
d_1-d_2	12.35	
d_4-d_5		7.2
d_7-d_8		4.85
Sums	12.35	12.15
	0.20	
× sin 60°	0.173	
÷9	0.019 = A_6	

thing, on the ordinates on which those analyses rest. Their good agreement is therefore to be regarded merely as a check on the two methods and the results of calculation.

EXAMPLE 13. ILLUSTRATING THE RESOLUTION OF A POWER CURVE INTO COMPONENTS

In this example the power curve considered in the preceding example (corresponding to the emf and current curves of Fig. 3) will be resolved into two components. Both systems of resolution,

TABLE 17

Comparison of Equations of Power Curves found by Analysis and by Calculation

	Power curve				Power components							
					ei_1				ei_2			
	Amplitude		Phase ψ		Amplitude		Phase ψ		Amplitude		Phase ψ	
	C	A	C	A	C	A	C	A	C	A	C	A
0	10.314	10.316			11.20	10.878						
2	21.848	21.843	78.8	78.8	(10.142)	10.142	46.6	46.8	(20.185)	20.185	91.4	91.9
4	3.307	3.242	37.1	37.1	1.730	1.524	26.3	26.3	2.448	2.422	43.8	42.4
6	0.962	0.972	0.4	− 0.2	0.715	0.885	51.2	50.1	0.824	0.966	7.6	8.4
8	0.022	0.032	19.4	21.4	0.144	0.103	31.8	30.5	0.145	0.102	10.4	33.6
10	0.353	0.340	15.1	15.0	0.272	0.144	11.5	13.1	0.216	0.218	19.5	20.0
12	0.753	0.750	22.7	22.6	0.583	0.556	26.2	26.5	0.522	0.496	18.9	19.3
14	1.091	1.084	10.1	10.1	1.125	1.058	10.2	10.3	0.065	0.077	9.6	14.9
16	0.960	0.976	1.1	1.1	0.685	0.604	20.5	20.5	0.753	0.839	3.7	3.6
18	0.082		3.7		0.007		2.4		0.075	0.012	3.9	1.6

treated on pages 614 and 615, will be employed, (*a*) that in which the emf. is resolved into two components, one in phase with the current, and the other 90° ahead of it, and (*b*) that in which the current is resolved into components, one along the emf vector and the other 90° behind it. The difference in the physical significance of these two methods has been already discussed (pp. 615 to 617).

The methods of calculation, outlined above (pp. 618 and 619), are illustrated in Tables 18 to 23, and should be understood more

clearly after an examination of the method of calculation of the power curve, given in the example above.

TABLE 18

Derivation of the Two Components of the Power Curve. Resolution of the Emf Curve and the Current Curve

m Order	$(\delta_m-\theta_m)$	sin $(\delta_m-\theta_m)$	cos $(\delta_m-\theta_m)$	E_m	E'_m	E''_m	I_m	I'_m	I''_m
	°								
1	65.3	0.9085	0.4179	33.217	13.880	30.178	29.627	12.380	−26.916
3	77.2	.9751	.2215	0.970	0.215	0.946	3.866	0.856	− 3.770
5	− 77.8	− .9774	.2113	1.702	.360	−1.663	0.459	.0972	0.448
7	−161.0	− .3256	− .9455	0.218	− .206	− .0710	.211	− .200	.0688
9	84.5	.9954	.0958	0.355	.0360	.353	.155	.0149	− .154
11	− 42.1	− .6704	.7420	0.205	.154	− .137	.163	.121	.109
13	79.8	.9842	.1771	1.251	.221	1.231	.214	.0379	− .210
15	− 6.5	− .1132	.9936	1.528	1.518	− .173	.188	.187	.0212
17	256.9	− .9740	− .2250	0.159	− .0358	− .155	.052	− .0117	.0517

We have first to find the components of the current and emf. just referred to. This is carried out in Table 18. This table contains in the first column the order of the harmonics, in the second the phase differences $(\delta_m—\theta_m)$ between the corresponding harmonics of current and emf (the phase difference is taken positive for a lagging current) and in the third and fourth columns the sines and cosines of these angles. The next columns give the harmonics E_m of the emf wave as found by analysis, and the components E'_m and E''_m, respectively, in phase with the current and 90° ahead of it. Similarly, in column 8 are given the harmonic amplitudes of the current, and in columns 9 and 10 the components I'_n and I''_n, respectively, in phase with the emf. and 90° behind it.

(*a*) **By resolution of the emf curve.**—The components of emf just found are used in Tables 19 and 20 to compute the equations of the power component curves e_1i and e_2i, according to the methods sketched on pages 618 and 619. In the first section of Table 19, which treats of the calculation of the curve e_1i, the products E'_mI_n are shown. Thus, $E'_5I_{13}=0.08$ is found in the column headed by I_{13}, and in the row corresponding to harmonics of order 5.

TABLE 19

Calculation of Equation of Curve of Power Component $e_p i$

(Resolution of Emf Curve)

Order	I_n	E'_m	δ_n	I_1	I_3	I_5	I_7	I_9	I_{11}	I_{13}	I_{15}	I_{17}
1	29.627	13.880	66°9	411.24	53.66	6.38	2.93	2.15	2.26	2.97	2.61	0.72
3	3.866	0.215	83.5	6.37	0.83	0.10	.045	.03	.035	.046	.04	.01
5	0.459	.360	131.0	10.65	1.39	.165	.08	.056	.06	.08	.07	.02
7	.211	−.206	107.1	−6.10	−0.80	−.095	−.04	−.03	−.034	−.04	−.04	−.01
9	.155	.0360	104.6	1.06	0.14	.016	.01	.01	.01	.01	.01	.00
11	.163	.154	169.7	4.57	.595	.07	.03	.02	.025	.03	.03	.01
13	.214	.221	129.1	6.55	.85	.10	.05	.03	.035	.05	.04	.01
15	.188	1.518	229.1	45.03	5.87	.70	.32	.24	.25	.325	.285	.08
17	.052	−.0358	308.0	−1.06	−.14	−.02	−.01	−.01	−.01	−.01	−.01	.00

CALCULATION OF TERM IN $12pt$

n	m	$E'_m I_n$	$(\delta_n+\delta_m)$	$\pm(\delta_m-\delta_n)$	Amplitudes M	Phase a	sin a	cos a	M sin a	M cos a
1	11	−4.57	236°6		−6.83	236.6	−.8348	−.5505	5.70	3.76
1	13	6.55		62.2	9.52	62.2	.8846	.4664	8.43	4.44
3	9	−0.14	188.1		−0.17	188.1	−.1409	−.9900	0.02	0.17
3	15	5.87		145.6	5.91	145.6	.5650	−.8251	3.34	−4.87
5	7	0.095	238.1		0.015	238.1	−.8490	−.5284	−0.01	−0.01
5	17	−0.02		177.0	0.00	177.0	.0523	−.9985		
7	5	−0.08	238.1							
9	3	−0.03	188.1					Sums=	17.48	3.49
11	1	−2.26	236.6					$N=\sqrt{17.48^2+3.49^2}=17.83$		
13	1	2.97		62.2				$\tan r=\frac{-3.49}{17.48}$		
15	3	0.04		145.6						
17	5	0.02		177.0				$r=348°.7$		

In the second part of the table is shown, in detail, the calculation of the term in $12pt$. Here those products $E'_m I_n$ enter, for which $(m+n)=12$ (in which case the product is given the negative sign), and those for which $(m-n)=12$, or $(n-m)=12$, the sign being positive in both the latter cases. With regard to the phase, we have as rules, that for $(m+n)=12$ the phase is $(\delta_n+\delta_m)$; for $(m-n)=12$ or $(n-m)=12$, the phase is $\pm(\delta_m-\delta_n)$, the sign being taken positive or negative according as m is greater or less than n. (These rules are exactly the same as in the calculation of the power curve.)

The rest of the calculation is, however, simpler than in the case of the power curve, since here the terms group themselves in pairs, for which the phase difference is the same. The ampli-

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TABLE 20

Calculation of the Equation of the Curve of the Wattless Component e_2i of the Power Curve

(Resolution of the Emf Curve)

Order	I_n	E_m''	δ_n	I_1	I_3	I_5	I_7	I_9	I_{11}	I_{13}	I_{15}	I_{17}
1	29.627	30.178	66°9	894.08	116.67	13.85	6.37	4.68	4.925	6.465	5.675	1.57
3	3.866	0.946	83.5	28.03	3.66	0.43	0.20	0.15	0.15	0.20	0.18	0.05
5	0.459	−1.663	131.0	−49.27	−6.425	−0.76	−0.35	−0.26	−0.27	−0.355	−0.31	−0.09
7	0.211	−0.0710	107.1	−2.11	−0.27	−0.03	−0.015	−0.01	−0.01	−0.01	−0.01	−0.00
9	0.155	0.353	104.6	10.46	1.36	0.16	0.07	0.055	0.06	0.075	0.07	0.02
11	0.163	−0.137	169.7	−4.06	−0.53	−0.06	−0.03	−0.02	−0.02	−0.03	−0.03	−0.01
13	0.214	1.231	129.1	36.32	4.76	0.56	0.26	0.19	0.20	0.26	0.23	0.06
15	0.188	−0.173	229.1	−5.155	−0.67	−0.08	−0.035	−0.03	−0.03	−0.04	−0.03	−0.01
17	0.052	0.155	308.0	−4.585	−0.60	−0.07	−0.03	−0.025	−0.025	−0.03	−0.03	−0.01

CALCULATION OF TERM IN 12*pt*

m	n	$E_m''I_n$	$(\delta_m+\delta_n)$	$\pm(\delta_m-\delta_n)$	Amplitudes M	Phases α	sin α	cos α	M sin α	M cos α
11	1	− 4.06	236°6		0.865	236.6	−0.8348	−.5505	− 0.72	− 0.48
1	11	4.925	"		−29.855	62.2	.8846	.4664	−26.41	−13.92
13	1	−36.32		62°2	1.51	188.1	− .1409	−.9900	− 0.21	− 1.49
1	13	6.465		"	0.85	145.6	.5650	−.8251	0.48	− 0.70
9	3	1.36	188.1		− 0.38	238.1	− .8490	−.5284	0.32	0.20
3	9	0.15	"		− 0.02	177.0	.0523	.9985		0.02
15	3	0.67		145.6						
3	15	0.18		"					−26.54	−16.37
7	5	− 0.03	238.1							
5	7	− 0.35	"							
17	5	0.07		177.0						
5	17	− 0.09		"						

$N=\sqrt{26.54^2+16.37^2}=31.18$

$\tan \gamma=\frac{-26.54}{-16.37}$

$\gamma=238°4$

CHECK ON RESOLUTION OF POWER CURVE TERM IN 12*pt*

Component	Amplitudes M	Phases α	sin α	cos α	M sin α	M cos α
Power$=e_1i$	17.83	348°7	−0.1959	0.9806	− 3.49	17.48
Wattless$=e_2i$	31.18	238.4	−0.8517	−0.5240	−26.54	−16.37
					−30.03	− 1.11

$N=\sqrt{30.03^2+1.11^2}=30.05$

$\tan \gamma=\frac{-30.03}{1.11}$

$\gamma=272°1$

For power curve
$N=30.12$
$\gamma=271°9$

tudes M thus grouped, together with their phases α, are given in columns 6 and 7. The sines and cosines of α follow, and the products $M \sin \alpha$ and $M \cos \alpha$. The method of derivation of the amplitude N and the phase γ from these products is evident from the table.

Table 20 treats of the calculation of the equation of the curve of $e_2 i$. The first part of the table shows the calculation of the products $E_m'' I_n$, the system of arrangement being the same as in the preceding tables. Then follows the detailed calculation of a single term, that in $12pt$. Referring to the equation

$$2 \sin (npt-\delta_n) \cos (mpt-\delta_m) = \sin [(m+n)pt-(\delta_m+\delta_n)] + \sin [(n-m)pt-(\delta_n-\delta_m)]$$

it is evident that those products for which $(m+n)=12$ enter without change of sign, as well as those for which $(n-m)=12$, while those for which $(m-n)=12$ are to be taken with the reversed sign. With regard to the phase, it is $(\delta_m+\delta_n)$ for $(m+n)=12$, $(\delta_n-\delta_m)$ for $(n-m)=12$, and $(\delta_m-\delta_n)$ for $(m-n)=12$.

Here, as in the preceding case, the terms group themselves in pairs which have the same phase. The calculation of the resultant amplitude N and the resultant phase γ is given in detail in the table.

In the last part of Table 20 is given a check on the derivation of the term in $12pt$ for the curves $e_1 i$ and $e_2 i$. The sum of these two terms should equal the term in $12pt$ in the equation of total power (Example 12). This condition is closely realized by the values calculated in Tables 19 and 20, as is here shown, the sum being $30.05 \sin (12pt-272^\circ.1)$ while the value calculated for the curve of total power in Example 12 is $30.12 \sin (12pt-271^\circ.9)$. This check is of great value, although, in case it is not realized in any given case, it gives no indication as to which of the curves $e_1 i$ and $e_2 i$ is in error.

(*b*) **By resolution of the Current—Curves ei_1 and ei_2.**—The method of calculation for this case is illustrated in Tables 21 and 22. These tables bear a close resemblance to Tables 19 and 20, respectively, and will be clear from what has been said regarding the arrangement of the latter. Here, as in the preceding case, the calculation has been carried through for the term in

$12pt$ and at the end of Table 22 is given a check on the correctness of the terms found in the curves of ei_1 and ei_2.

TABLE 21

Calculation of the Equation of the Curve ei_1. Power Component Resolving the Current

Order	E_m	I_n'	θ_m	E_1	E_3	E_5	E_7	E_9	E_{11}	E_{13}	E_{15}	E_{17}
			°									
1	33.217	12.380	1.6	411.23	12.01	21.07	2.70	4.40	2.54	15.49	18.92	1.97
3	0.970	0.856	6.3	28.43	0.83	1.46	0.19	0.30	0.175	1.07	1.31	0.135
5	1.702	0.0972	208.8	3.23	0.095	0.165	0.02	0.035	0.02	0.12	0.15	0.015
7	0.218	−0.200	268.1	−6.64	−0.195	−0.34	−0.04	−0.07	−0.04	−0.25	−0.31	−0.03
9	0.355	0.0149	20.1	0.50	0.01	0.025	0.00	0.01	0.00	0.02	0.02	0.00
11	0.205	0.121	211.8	4.02	0.12	0.205	0.03	0.04	0.025	0.15	0.185	0.02
13	1.251	0.0379	49.3	1.26	0.04	0.065	0.01	0.01	0.01	0.05	0.06	0.01
15	1.528	0.187	235.6	6.215	0.18	0.32	0.04	0.07	0.04	0.23	0.29	0.03
17	0.159	−0.0117	51.1	−0.39	−0.01	−0.02	−0.00	−0.00	−0.00	−0.015	−0.02	−0.00

CALCULATION OF TERM IN $12pt$

m	n	E_mI_n'	$(\theta_m+\theta_n)$	$\pm(\theta_m-\theta_n)$	Amplitudes M	Phases α	sin α	cos α	M sin α	M cos α
			°			°				
1	11	− 4.02	213.4		− 6.56	213.4	−.5505	−.8348	3.61	5.47
11	1	− 2.54	"	°	16.75	47.7	.7396	.6730	12.39	11.27
1	13	1.26		47.7	− 0.31	26.4	.4446	.8957	− 0.14	− 0.28
13	1	15.49		"	1.49	229.3	−.7581	−.6521	− 1.13	− 0.97
3	9	− 0.01	26.4		0.32	476.9	.8918	−.4524	0.29	− 0.14
9	3	− 0.30	"		− 0.005	−157.7	−.3795	−.9252		
3	15	0.18		229.3					15.02	15.35
15	3	1.31		"						
5	7	0.34	476.9							
7	5	− 0.02	"							
5	17	− 0.02		−157.7						
17	5	0.015		"						

$$M=\sqrt{15.02^2+15.35^2}=21.48$$

$$\tan\gamma=\frac{-15.35}{15.02}$$

$$\gamma=314^{\circ}.3$$

For comparison, the equations of the four curves corresponding to the resolution of the power curve according to these two methods are presented in Table 23.

It will be noticed that the two methods lead to quite different curves. The latter part of the table contains the checks on the curves; that is, it gives an idea of how closely the sums of the two components in each case approaches the value of the power

TABLE 22

Calculation of the Equation of the Curve ei_2. Wattless Component Resolving Current

Order	E_m	I_n''	θ_m	E_1	E_3	E_5	E_7	E_9	E_{11}	E_{13}	E_{15}	E_{17}
1	33. 217	−26. 916	1. 6	−894. 08	−26. 11	−45. 81	−5. 87	−9. 56	−5. 52	−33. 68	−41. 13	−4. 285
3	0. 970	− 3. 770	6. 3	−125. 23	− 3. 655	− 6. 415	−0. 82	−1. 34	−0. 77	− 4. 72	− 5. 77	−0. 60
5	1. 702	0. 448	208. 8	14. 82	0. 435	0. 76	0. 10	0. 16	0. 09	0. 56	0. 685	0. 07
7	0. 218	0. 0688	268. 1	2. 28	0. 07	0. 12	0. 015	0. 025	0. 015	0. 085	0. 105	0. 01
9	0. 355	− 0. 154	20. 1	− 5. 12	− 0. 15	− 0. 26	−0. 03	−0. 055	−0. 03	− 0. 19	− 0. 235	−0. 025
11	0. 205	0. 109	211. 8	3. 62	0. 105	0. 185	0. 02	0. 04	0. 02	0. 14	0. 17	0. 02
13	1. 251	− 0. 210	49. 3	− 6. 975	− 0. 20	− 0. 36	−0. 05	−0. 075	−0. 04	−0. 26	− 0. 32	−0. 03
15	1. 528	0. 0212	235. 6	0. 70	0. 02	0. 04	0. 00	0. 01	0. 00	0. 025	0. 03	0. 00
17	0. 159	0. 0517	51. 1	1. 72	0. 05	0. 09	0. 01	0. 02	0. 01	0. 065	0. 08	0. 01

CALCULATION OF TERM IN 12*pt*

m	n	E_mI_n''	$(\theta_m+\theta_n)$	$\pm(\theta_m-\theta_n)$	Amplitudes M	Phases α	sin α	cos α	M sin α	M cos α
			°			°				
1	11	3. 62	213. 4		− 1. 90	213. 4	−0. 5505	−0. 8348	1. 05	1. 59
11	1	− 5. 52	″	°	−26. 705	47. 7	. 7396	. 6730	−19. 75	−17. 97
1	13	6. 975		47. 7	− 1. 49	26. 4	. 4446	. 8957	− 0. 66	− 1. 34
13	1	−33. 68		″	− 5. 79	229. 3	− . 7581	− . 6521	4. 39	3. 78
3	9	− 0. 15	26. 4		0. 22	476. 9	. 8918	− . 4524	0. 20	− 0. 10
9	3	− 1. 34	″		− 0. 02	−157. 7	− . 3795	− . 9252	0. 01	0. 02
3	15	− 0. 02		229. 3						
15	3	− 5. 77							−14. 76	−14. 02
5	7	0. 12	476. 9							
7	5	0. 10	″							
5	17	− 0. 09		−157. 7						
17	5	0. 07		″						

$$N=\sqrt{14.76^2+14.02^2}=20.33$$

$$\tan\gamma=\frac{-14.76}{-14.02}$$

$$\gamma=226.^\circ5$$

CHECK ON RESOLUTION OF POWER CURVE TERM IN 12*pt*

Component	Amplitudes M	Phases α	sin α	cos α	M sin α	M cos α
Power	21. 48	314. 3	−0. 7157	0. 6984	−15. 35	15. 02
Wattless	20. 33	226. 5	−0. 7254	−0. 6884	−14. 76	−14. 02
					−30. 11	1. 00

From power curve
N=30. 12
$\gamma=271.^\circ9$

$$N=\sqrt{30.11^2+1.00^2}=30.12$$

$$\tan\gamma=\frac{-30.11}{1.00}$$

$$\gamma=271.^\circ9$$

curve. The agreement shown in the table is to be regarded as satisfactory, considering the number of significant figures retained in the calculations.

It is of interest to compare these equations with those found by analysis of the curves *C* and *D* of Fig. 3. These were determined experimentally, as described in the introduction, the power attachment of the curve tracer having been set successively to the ordinates of the curves i_1 and i_2 (determined by resolving the analyzed current wave), while the emf of the dynamo was balanced on the curve tracer solenoid. These curves give, therefore, the instantaneous values of ei_1 and ei_2, respectively.

TABLE 23

Comparison of Equations of Components of Power Curve both Resolving the Emf and the Current

Order	e_1i		e_2i		ei_1		ei_2		Checks				Power curve	
									e_1i+e_2i		ei_1+ei_2			
	Amp.	Phase	Amp.	Phase	Amp.	Phase	Amp.	Phase	Amp.	Phase	Amp.	Phase	Amp.	Phase
		°		°		°		°		°		°		°
2	441.7	231.0	858.8	128.1	373.05	93.2	788.23	182.8	873.4	157.6	874.0	157.6	873.9	157.6
4	60.9	256.3	161.9	127.3	63.68	105.1	96.56	175.4	132.4	148.3	132.5	148.4	132.3	148.3
6	15.10	282.8	38.76	25.3	26.32	307.2	32.19	45.4	38.43	2.8	38.52	2.9	38.46	2.6
8	4.07	350.1	4.79	167.8	5.30	254.8	5.64	83.1	0.75	155.4	0.82	157.8	0.88	155.2
10	7.76	349.0	21.61	157.5	10.03	115.4	8.42	195.4	14.08	151.6	14.17	151.2	14.13	151.4
12	17.83	348.7	31.18	238.4	21.48	314.3	20.33	226°5	30.05	272.1	30.12	271.9	30.12	271.9
14	40.10	63.5	52.47	189.3	41.47	143.2	2.54	113.4	43.57	141.0	43.70	141.5	43.66	141.2
16	48.06	24.6	10.91	229.2	25.22	328.5	29.41	59.2	38.45	17.8	38.49	18.3	38.38	18.2
18	5.88	38.4	3.39	191.6	0.26	43.5	2.93	69.4	3.24	66.7	3.16	67.3	3.26	67.1

The experimental curves were analyzed by Schedule 3, Appendix B, and the results obtained will be found in Table 17 along with the calculated values given in Table 23, the amplitudes of the latter having been reduced by a factor so as to agree with the scale of the curves. The calculated values are indicated by the letter *C*, those found by analysis by the letter *A*. The phases are all reduced to the scale of the fundamental of the emf. and

current waves. The agreement of the curves found by the two methods is probably as good as could be expected.

No claim is made to anything approaching completeness in the list of applications here considered. Others will, no doubt, occur to readers engaged in this kind of work, and modifications of the procedure may have to be made to fit special cases. However, the author hopes that the treatment here given will not only be of material service in lightening the labor of ordinary routine calculations, but may be broad enough to cover the general methods which must be followed in attacking special problems.

VII. SUMMARY

1. The equations of Fourier for the resolution of a periodic curve into a series of component sine curves, although furnishing a complete solution of the problem, suffer under the disadvantage that, in making numerical calculations, a large number of products have to be formed. Consequently, other less satisfactory methods have been employed to shorten the calculation. By grouping similar terms, Runge has so simplified the use of the Fourier equations as to greatly reduce the labor of calculation. The object of this paper is to stimulate a more general employment of Runge's method by the systematic arrangement of the calculations and the use of multiplication tables such as to render inconsiderable the time and labor necessary for the precise analysis of alternating current curves.

2. The derivation of Fourier's equations for the case of a finite number of odd harmonics is sketched briefly, using the treatment in Byerly's "Fourier's Series and Spherical Harmonics," and this is followed by a development of the simplifications introduced by Runge.

3. The arrangement of the calculation is given in detail for three special cases, namely, for 6, 12, and 18 measured ordinates per half cycle, together with check equations which facilitate the location of numerical errors. The considerations which determine the choice of the proper schedule of analysis in any given case are treated at some length.

4. The complete calculation of the amplitude and phases of the component waves of experimentally determined alternating current curves is carried through, using each of the three schemes of analysis. Further examples are given in connection with what has been previously enunciated with respect to the choice of schedule and to illustrate the accuracy of the results.

5. The latter part of the paper is devoted to the consideration of a few practical applications, such as the calculation of average and effective values, average power, and equations of power curves, together with a full illustration of the principles involved in numerical examples.

6. In an appendix are given multiplication tables, from which may be taken the products which have to be formed in the use of the three analysis schedules. Further analysis schedules are added for curves in which even harmonics are present, which schedules may therefore be used for the analysis of power curves.

WASHINGTON, May 15, 1913.

APPENDIXES

APPENDIX A

TABLE 1

Multiplication Table for Use with the Six-Point and Eighteen-Point Analysis Schedules

	sin 10°	sin 20°	sin 30°	sin 40°	sin 50°	sin 60°	sin 70°	sin 80°
1	0.1736	0.3420	0.5000	0.6428	0.7660	0.8660	0.9397	0.9848
2	0.3473	0.6840	1.000	1.2855	1.532	1.732	1.879	1.970
3	0.5209	1.026	1.500	1.928	2.298	2.598	2.819	2.954
4	0.6946	1.368	2.000	2.571	3.064	3.464	3.759	3.939
5	0.8682	1.710	2.500	3.214	3.830	4.330	4.6985	4.924
6	1.042	2.052	3.000	3.857	4.596	5.196	5.638	5 909
7	1.2155	2.394	3.500	4.4995	5.362	6.062	6.578	6.894
8	1.389	2 736	4.000	5.142	6.128	6.928	7.518	7.878
9	1.563	3.078	4.500	5.785	6.894	7.794	8.457	8.863
10	1.7365	3.420	5.000	6.428	7.660	8.660	9.397	9.848
11	1.910	3.762	5.500	7.0705	8.4265	9.526	10.337	10.833
12	2.084	4.104	6.000	7.713	9.1925	10.392	11.276	11.818
13	2.2575	4.446	6.500	8.356	9.9585	11.258	12.216	12.802
14	2.431	4.788	7.000	8.999	10.7245	12.124	13.156	13.787
15	2.605	5.130	7.500	9.642	11.491	12.990	14.0955	14.772
16	2.778	5.472	8.000	10.2845	12.257	13.856	15.035	15.757
17	2.952	5.814	8.500	10.927	13.023	14.722	15.975	16.742
18	3.126	6.156	9.000	11.570	13.789	15.588	16.915	17.726
19	3.299	6.498	9.500	12.213	14.555	16.454	17.854	18.711
20	3.473	6.8405	10.000	12.856	15.321	17.320	18.794	19.696
21	3.647	7.1825	10.500	13.498	16.087	18.1865	19.734	20.681
22	3.820	7.5245	11.000	14.141	16.853	19.0525	20.673	21.666
23	3.994	7.8665	11.500	14.784	17.619	19.9185	21.613	22.650
24	4.168	8.2085	12.000	15.427	18.385	20.7845	22.553	23.635
25	4.341	8.5505	12.500	16.0695	19.151	21.6505	23.4925	24.620
26	4.515	8.8925	13.000	16.712	19.917	22.5165	24.432	25.605
27	4.6885	9.2345	13.500	17.355	20.683	23.3825	25.372	26.590
28	4.862	9.5765	14.000	17.998	21.449	24.2485	26.312	27.574
29	5.036	9.9185	14.500	18.641	22.215	25.1145	27.251	28.559
30	5.2095	10.027	15.000	19.283	22.981	25.981	28.191	29.544
31	5.383	10.603	15.500	19.926	23.757	26.847	29.131	30.529
32	5.557	10.945	16.000	20.569	24.513	27.713	30.070	31.514
33	5.7305	11.287	16.500	21.212	25.279	28.579	31.010	32.498
34	5.904	11.629	17.000	21.8545	26.045	29.445	31.950	33.483
35	6.078	11.971	17.500	22.497	26.811	30.311	32.8895	34.468
36	6.251	12.313	18.000	23.140	27.577	31.177	33.829	35.453
37	6.425	12.655	18.500	23.783	28.3435	32.043	34.769	36.438
38	6.599	12.997	19.000	24.426	29.1095	32.909	35.709	37.422
39	6.772	13.339	19.500	25.068	29.8755	33.775	36.648	38.407
40	6.946	13.681	20.000	25.711	30.642	34.641	37.588	39.392
41	7.120	14.023	20.500	26.354	31.408	35.507	38.528	40.377
42	7.293	14.365	21.000	26.997	32.174	36.373	39.467	41.362
43	7.467	14.707	21.500	27.6395	32.940	37.239	40.407	42.346
44	7.641	15.049	22.000	28.282	33.706	38.105	41.347	43.331
45	7.814	15.391	22.500	28.925	34.472	38.971	42.2865	44.316
46	7.988	15.733	23.000	29.568	35.238	39.837	43.226	45.301
47	8.1615	16.075	23.500	30.211	36.004	40.703	44.166	46.286
48	8.335	16.417	24.000	30.8535	36.770	41.569	45.106	47.270
49	8.509	16.759	24.500	31.496	37.536	42.435	46.045	48.255
50	8.6825	17.101	25.000	32.139	38.302	43.301	46.985	49.240

TABLE 1—Continued

	sin 10°	sin 20°	sin 30°	sin 40°	sin 50°	sin 60°	sin 70°	sin 80°
51	8.856	17.443	25.500	32.782	39.068	44.167	47.925	50.225
52	9.030	17.785	26.000	33.4245	39.834	45.033	48.864	51.210
53	9.2035	18.127	26.500	34.067	40.600	45.899	49.804	52.194
54	9.377	18.469	27.000	34.710	41.366	46.765	50.744	53.179
55	9.551	18.811	27.500	35.353	42.132	47.631	51.6835	54.164
56	9.724	19.153	28.000	35.996	42.898	48.497	52.623	55.149
57	9.898	19.495	28.500	36.6385	43.664	49.363	53.563	56.134
58	10.072	19.837	29.000	37.281	44.430	50.229	54.503	57.118
59	10.245	20.180	29.500	37.924	45.196	51.095	55.442	58.103
60	10.419	20.521	30.000	38.567	45.962	51.961	56.382	59.088
61	10.593	20.863	30.500	39.210	46.7285	52.827	57.322	60.073
62	10.766	21.205	31.000	39.852	47.4945	53.693	58.261	61.058
63	10.940	21.547	31.500	40.495	48.2605	54.559	59.201	62.042
64	11.114	21.889	32.000	41.138	49.0265	55.425	60.141	63.027
65	11.287	22.231	32.500	41.781	49.793	56.291	61.0805	64.012
66	11.461	22.573	33.000	42.4235	50.559	57.157	62.020	64.997
67	11.6345	22.915	33.500	43.066	51.325	58.023	62.960	65.982
68	11.808	23.257	34.000	43.709	52.091	58.889	63.900	66.966
69	11.982	23.599	34.500	44.352	52.857	59.755	64.839	67.951
70	12.1555	23.9415	35.000	44.995	53.623	60.621	65.779	68.936
71	12.329	24.2835	35.500	45.637	54.389	61.487	66.719	69.921
72	12.503	24.6255	36.000	46.280	55.155	62.353	67.658	70.906
73	12.676	24.9675	36.500	46.923	55.921	63.2195	68.598	71.890
74	12.850	25.3095	37.000	47.566	56.687	64.0855	69.538	72.875
75	13.024	25.6515	37.500	48.2085	57.453	64.9515	70.4775	73.860
76	13.197	25.9935	38.000	48.851	58.219	65.8175	71.417	74.845
77	13.371	26.3355	38.500	49.494	58.984	66.6835	72.357	75.830
78	13.545	26.6775	39.000	50.137	59.751	67.5495	73.297	76.814
79	13.718	27.0195	39.500	50.780	60.517	68.420	74.236	77.799
80	13.892	27.362	40.000	51.422	61.283	69.282	75.176	78.784
81	14.066	27.704	40.500	52.065	62.049	70.148	76.116	79.769
82	14.239	28.046	41.000	52.708	62.815	71.014	77.055	80.754
83	14.413	28.388	41.500	53.351	63.581	71.880	77.995	81.738
84	14.587	28.730	42.000	53.9935	64.347	72.746	78.935	82.723
85	14.760	29.072	42.500	54.636	65.113	73.612	79.8745	83.708
86	14.934	29.414	43.000	55.279	65.879	74.478	80.814	84.693
87	15.1075	29.756	43.500	55.922	66.6455	75.344	81.754	85.678
88	15.281	30.098	44.000	56.565	67.4115	76.210	82.694	86.662
89	15.455	30.440	44.500	57.207	68.1775	77.076	83.633	87.647
90	15.6285	30.782	45.000	57.850	68.944	77.942	84.573	88.632
91	15.802	31.124	45.500	58.493	69.710	78.808	85.513	89.617
92	15.976	31.466	46.000	59.136	70.476	79.674	86.452	90.602
93	16.1495	31.808	46.500	59.7785	71.242	80.540	87.392	91.586
94	16.323	32.150	47.000	60.421	72.008	81.406	88.332	92.571
95	16.497	32.492	47.500	61.064	72.774	82.272	89.2715	93.556
96	16.670	32.834	48.000	61.707	73.540	83.138	90.211	94.541
97	16.844	33.176	48.500	62.350	74.306	84.004	91.151	95.526
98	17.018	33.518	49.000	62.992	75.072	84.870	92.091	96.510
99	17.191	33.860	49.500	63.635	75.838	85.736	93.030	97.495
100	17.365	34.202	50.000	64.278	76.604	86.602	93.970	98.480

TABLE 2

Multiplication Table for Use with the 12-Point Analysis Schedule

	sin 15°	sin 45°	sin 75°		sin 15°	sin 45°	sin 75°
1	0.2588	0.7071	0.9659	51	13.200	36.063	49.262
2	0.5176	1.414	1.932	52	13.459	36.770	50.228
3	0.7765	2.121	2.898	53	13.7175	37.477	51.194
4	1.035	2.828	3.864	54	13.976	38.184	52.160
5	1.294	3.5355	4.830	55	14.235	38.891	53.126
6	1.553	4.243	5.796	56	14.494	39.598	54.092
7	1.812	4.950	6.7615	57	14.753	40.305	55.058
8	2.071	5.657	7.727	58	15.012	41.012	56.024
9	2.329	6.364	8.693	59	15.270	41.7195	56.990
10	2.588	7.071	9.659	60	15.529	42.427	57.956
11	2.847	7.778	10.625	61	15.788	43.134	58.922
12	3.106	8.485	11.591	62	16.047	43.841	59.888
13	3.365	9.192	12.557	63	16.306	44.548	60.854
14	3.6235	9.8995	13.523	64	16.5645	45.255	61.8195
15	3.882	10.607	14.489	65	16.823	45.962	62.785
16	4.141	11.314	15.455	66	17.082	46.669	63.751
17	4.400	12.021	16.421	67	17.341	47.336	64.717
18	4.659	12.728	17.387	68	17.600	48.0835	65.683
19	4.918	13.435	18.353	69	17.859	48.791	66.649
20	5.176	14.142	19.319	70	18.117	49.498	67.615
21	5.435	14.849	20.2845	71	18.376	50.205	68.581
22	5.694	15.556	21.2505	72	18.635	50.912	69.547
23	5.953	16.2635	22.216	73	18.894	51.619	70.513
24	6.212	16.971	23.182	74	19.153	52.326	71.479
25	6.4705	17.678	24.148	75	19.4115	53.033	72.445
26	6.729	18.385	25.114	76	19.670	53.740	73.411
27	6.988	19.092	26.080	77	19.929	54.4475	74.377
28	7.247	19.799	27.046	78	20.188	55.155	75.3425
29	7.506	20.506	28.012	79	20.447	55.862	76.3085
30	7.765	21.213	28.978	80	20.706	56.569	77.274
31	8.023	21.920	29.944	81	20.964	57.276	78.240
32	8.282	22.6275	30.910	82	21.223	57.983	79.206
33	8.541	23.335	31.876	83	21.482	58.690	80.172
34	8.800	24.042	32.842	84	21.741	59.397	81.138
35	9.059	24.749	33.8075	85	22.000	60.104	82.104
36	9.3175	25.456	34.7735	86	22.2585	60.8115	83.070
37	9.576	26.163	35.739	87	22.517	61.519	84.036
38	9.835	26.870	36.705	88	22.776	62.226	85.002
39	10.094	27.577	37.671	89	23.035	62.933	85.968
40	10.353	28.284	38.637	90	23.294	63.640	86.934
41	10.612	28.9915	39.603	91	23.553	64.347	87.900
42	10.870	29.699	40.569	92	23.811	65.054	88.866
43	11.129	30.406	41.535	93	24.070	65.761	89.8315
44	11.388	31.113	42.501	94	24.329	66.468	90.797
45	11.647	31.820	43.467	95	24.588	67.175	91.763
46	11.906	32.527	44.433	96	24.847	67.883	92.729
47	12.1645	33.234	45.399	97	25.1055	68.590	93.695
48	12.423	33.941	46.365	98	25.364	69.297	94.661
49	12.682	34.648	47.331	99	25.623	70.004	95.627
50	12.941	35.3555	48.2965	100	25.882	70.711	96.593

APPENDIX B

ANALYSIS OF CURVES CONTAINING EVEN HARMONICS AND A CONSTANT TERM. (POWER CURVES)

In this section is considered the analysis of curves which are capable of being represented by an equation of the form

$$y = B_0 + B_1 \cos pt + B_2 \cos 2pt + B_3 \cos 3pt + \ . \ . \ . \ + B_n \cos npt$$
$$+ A_1 \sin pt + A_2 \sin 2pt + A_3 \sin 3pt + \ . \ . \ . \ + A_n \sin npt.$$

The fundamental ordinates are supposed to be $2n$ in number, equally spaced over the whole cycle. (In the analysis of current and electromotive force curves, we have constantly used fundamental ordinates equally spaced over the *half* cycle.)

What follows has been arranged with special reference to the analysis of power curves. These are represented by equations like that above, the frequency of the fundamental of the power curve being twice that of the fundamental of the emf. or current, and one complete cycle corresponds to a half cycle of the emf. or current. If, therefore, the analysis is being made of the emf. and current, it is only necessary for obtaining the equation of the power curve also, to take the products of the corresponding fundamental ordinates used for the analysis of the current and emf. and perform on these the operations indicated in the analysis schedules below. This does not require that the power curve shall have been drawn.

The schemes of analysis for power curves here given have been selected to correspond with those already given for current and emf. curves. Thus, if these have been analyzed by means of the 12-point schedule, Table 2, then the power curve will be obtained using the 12-point schedule in Table 2 below, etc. Such an analysis does not give as complete an equation for the power as may be obtained by taking the product of the equations of emf. and current (p. 612). For example, the product of the equations of emf. and current, obtained by the use of the 18-point schedule will contain harmonics of 34 times the frequency of the fundamental, while the term of highest order given by the analysis of the power curve, Table 3 below, has a frequency only 18 times that of the fundamental. In all cases, however, except those

where the higher harmonics in the current and emf. waves are abnormally prominent, all terms beyond those included in the analysis will be negligible.

It may be shown as was done for equation (2) that the coefficients in the above equation are given by the following equations

$$A_k = \frac{1}{n}\sum_{0}^{2n-1} y_m \sin km\frac{\pi}{n}$$

$$B_k = \frac{1}{n}\sum_{0}^{2n-1} y_m \cos km\frac{\pi}{n} \qquad k = 1, 2, \ldots (n-1)$$

$$B_0 = \frac{1}{2n}\sum_{0}^{2n-1} y_m \qquad B_n = \frac{1}{2n}\sum_{0}^{2n-1} (-1)^m y_m$$

The analysis from the $2n$ fundamental ordinates does not allow of the determination of A_n. This can, however, be easily found by measuring another set of ordinates z_m, situated half way between those which are used in the analysis, and substituting them in the equation

$$A_n = \frac{1}{2n}\sum_{0}^{2n-1} (-1)^m z_m$$

The following schedules for facilitating calculations with these equations, for the cases of 6, 12, and 18 measured ordinates, respectively, follow the method of Runge, as in the preceding analysis schedules in Tables 1, 2, and 3, The same multiplication table (Appendix A) may be used as in those cases. For an example of a numerical calculation, see example 12*b*.

SCHEDULE 1

Analysis of a Curve, Involving Even Harmonics and a Constant Term, from Six Ordinates

Arrange the measured ordinates according to the following scheme and take the sums and differences indicated:

	y_o	y_1	y_2	y_3	
		y_5	y_4		
Sums	s_o	s_1	s_2	s_3	$s_o+s_2=\Sigma_o$
Diffs	d_o	d_1	d_2	d_3	$s_1+s_3=\Sigma_1$

(the sum $s_0=d_0=y_0$, but this nomenclature is adhered to for uniformity.)
The coefficients are those given in the schedule below, the arrangement being the same as in the previous schedules—Table 1, 2, and 3:

	sine terms		cosine terms			
	A_1 and A_2		B_1 and B_2		B_o and B_2	
sin 30°			$-s_2$	s_1		
sin 60°	s_1	s_2				
sin 90°			s_o	$-s_3$	Σ_o	Σ_1
Sums	S_o'	S_e'	S_o''	S_e''	S_o'''	S_e'''
		$A_1=\frac{S_o'+S_e'}{3}$		$B_1=\frac{S_o''+S_e''}{3}$		$B_o=\frac{S_o'''+S_e'''}{6}$
		$A_2=\frac{S_o'-S_e'}{3}$		$B_2=\frac{S_o''-S_e''}{3}$		$B_3=\frac{S_o'''-S_e'''}{6}$

CHECKS

$$s_o=\ (B_o+B_3)+(B_1+B_2)$$
$$s_2=2(B_o+B_3)-(B_1+B_2)$$
$$s_o+s_2=3(B_o+B_3) \qquad 2s_o-s_2=3(B_1+B_2)$$
$$s_1=2(B_o-B_3)+(B_1-B_2$$
$$s_3=\ (B_o-B_3)-(B_1-B_2)$$
$$s_1+s_3=3(B_o+B_3) \qquad s_1-2s_3=3(B_1-B_2)$$
$$d_1=2(A_1+A_2)\sin 60°$$
$$d_2=2(A_1-A_2)\sin 60°$$

The first equation checks the sums of the *B's*. If it is not fulfilled, equations may be used in which these sums appear singly. The same procedure is to be adopted with the differences of the *B's*. The sum and difference of the *A's* occur singly. These checks serve as a control on the values of S_o and S_e.

SCHEDULE 2

Analysis of a Curve, Involving Even Harmonics and a Constant Term, from Twelve Ordinates

MEASURED ORDINATES

	y_0 y_1 y_2 y_3 y_4 y_5 y_6 y_{11} y_{10} y_9 y_8 y_7	s_0 s_1 s_2 s_3 s_6 s_5 s_4	d_0 d_1 d_2 d_3 d_6 d_5 d_4
Sums Diffs.	s_0 s_1 s_2 s_3 s_4 s_5 s_6 d_0 d_1 d_2 d_3 d_4 d_5 d_6	Sums Σ_0 Σ_1 Σ_2 Σ_3 Diffs. Δ_0 Δ_1 Δ_2 Δ_3	σ_0 σ_1 σ_2 σ_3 Sums δ_0 δ_1 δ_2 δ_3 Diffs.

	sine terms		
	A_1 and A_5	A_2 and A_4	A_3
sin 30° sin 60° sin 90°	σ_1 σ_2 σ_3	δ_1 δ_2	$(\sigma_1-\sigma_3)$
Sums	S_o' S_e' $A_1=\frac{S_o'+S_e'}{6}$ $A_5=\frac{S_o'-S_e'}{6}$	S_o'' S_e'' $A_2=\frac{S_o''+S_e''}{6}$ $A_4=\frac{S_o''-S_e''}{6}$	S_o''' $A_3=\frac{S_o'''}{6}$

	cosine terms			
	B_1 and B_5	B_2 and B_4	B_3	B_0 and B_6
sin 30° sin 60° sin 90°	Δ_2 Δ_1 Δ_0	$-\Sigma_2$ Σ_1 Σ_0 $-\Sigma_3$	$(\Delta_0-\Delta_2)$	$(\Sigma_0+\Sigma_2)$ $(\Sigma_1+\Sigma_3)$
Sums	D_o' D_e' $B_1=\frac{D_o'+D_e'}{6}$ $B_5=\frac{D_o'-D_e'}{6}$	D_o'' D_e'' $B_2=\frac{D_o''+D_e''}{6}$ $B_4=\frac{D_o''-D_e''}{6}$	D_o''' $B_3=\frac{D_o'''}{6}$	D_o'''' D_e'''' $B_0=\frac{D_o''''+D_e''''}{12}$ $B_6=\frac{D_o''''-D_e''''}{12}$

CHECKS

$\Sigma_0=(B_0+B_6)+(B_1+B_5)+(B_2+B_4)+B_3$

$\Sigma_0=2[(B_0+B_6)+(B_2+B_4)]$

$\Delta_0=2[(B_1+B_5)+B_3]$

$\Sigma_1=4\,(B_0-B_6)+2(B_2-B_4)$

$\Delta_1=4\,(B_1-B_5)\sin 60°$

$\sigma_1=2\,(A_1+A_5)+4A_3$

$\delta_1=4\,(A_2+A_4)\sin 60°$

$\sigma_2=4\,(A_1-A_5)\sin 60°$

$\delta_2=4\,(A_2-A_4)\sin 60°$

SCHEDULE 3

Analysis of a Curve Involving Even Harmonics and a Constant Term from Eighteen Ordinates

MEASURED ORDINATES

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
		y_{17}	y_{16}	y_{15}	y_{14}	y_{13}	y_{12}	y_{11}	y_{10}	
Sums	s_0	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9
Diffs	d_0	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9

$d_1-d_5+d_7=\triangle_1$

$d_2-d_4+d_8=\triangle_2$

$s_0+s_6=\Sigma_0$

$s_3+s_9=\Sigma_3$

$s_1+s_5+s_7=\Sigma_1$

$s_2+s_4+s_8=\Sigma_2$

$\Sigma_0+\Sigma_2=\delta_0$

$\Sigma_1+\Sigma_3=\delta_1$

	Sine terms			
	A_1 and A_8	A_2 and A_7	A_3 and A_6	A_4 and A_5
sin 20°	d_1 d_8	$-d_5$ d_4		$-d_7$ d_2
sin 40°	d_7 d_2	d_1 $-d_8$		d_5 $-d_4$
sin 60°	d_3 d_6	d_3 $-d_6$	$\triangle_1$ $\triangle_2$	$-d_3$ d_6
sin 80°	d_5 d_4	$-d_7$ d_2		d_1 $-d_8$
Sums	D_o D_e	D_o' D_e'	D_o'' D_e''	D_o''' D_e'''
	$A_1=\frac{D_o+D_e}{9}$	$A_2=\frac{D_o'+D_e'}{9}$	$A_3=\frac{D_o''+D_e''}{9}$	$A_4=\frac{D_o'''+D_e'''}{9}$
	$A_8=\frac{D_o-D_e}{9}$	$A_7=\frac{D_o'-D_e'}{9}$	$A_6=\frac{D_o''-D_e''}{9}$	$A_5=\frac{D_o'''-D_e'''}{9}$

	Cosine terms				
	B_1 and B_8	B_2 and B_7	B_3 and B_6	B_4 and B_5	B_0 and B_9
sin 10°	s_4 $-s_5$	s_2 s_7		s_8 s_1	
sin 30°	$-s_6$ s_3	$-s_6$ $-s_3$	$-\Sigma_2$ Σ_1	$-s_6$ $-s_3$	
sin 50°	s_2 $-s_7$	s_8 s_1		s_4 s_5	
sin 70°	$-s_8$ s_1	$-s_4$ $-s_5$		$-s_2$ $-s_7$	
sin 90°	s_0 $-s_9$	s_0 s_9	Σ_0 $-\Sigma_3$	s_0 s_9	δ_0 δ_1
Sums	S_o S_e	S_o' S_e'	S_o'' S_e''	S_o''' S_e'''	S_o'''' S_e''''
	$B_1=\frac{S_o+S_e}{9}$	$B_2=\frac{S_o'+S_e'}{9}$	$B_3=\frac{S_o''+S_e''}{9}$	$B_4=\frac{S_o'''+S_e'''}{9}$	$B_o=\frac{S_o''''+S_e''''}{18}$
	$B_8=\frac{S_o-S_e}{9}$	$B_7=\frac{S_o'-S_e'}{9}$	$B_6=\frac{S_o''-S_e''}{9}$	$B_5=\frac{S_o'''-S_e'''}{9}$	$B_9=\frac{S_o''''-S_e''''}{18}$

CHECKS

$s_0=(B_o+B_9)+(B_1+B_8)+(B_2+B_7)+(B_3+B_6)+(B_4+B_5)$

$s_9=(B_o-B_9)-(B_1-B_8)+(B_2-B_7)-(B_3-B_6)+(B_4-B_5)$

$d_3=2[(A_1+A_8)+(A_2+A_7)-(A_4+A_5)]\sin 60°$

$d_6=2[(A_1-A_8)-(A_2-A_7)+(A_4-A_5)]\sin 60°$